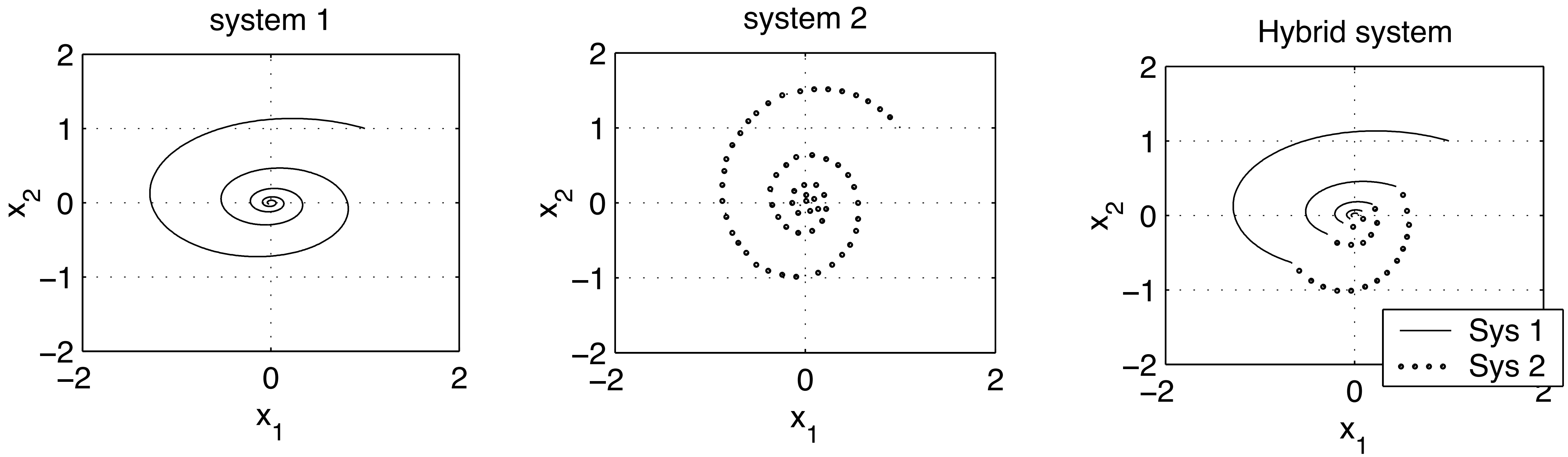
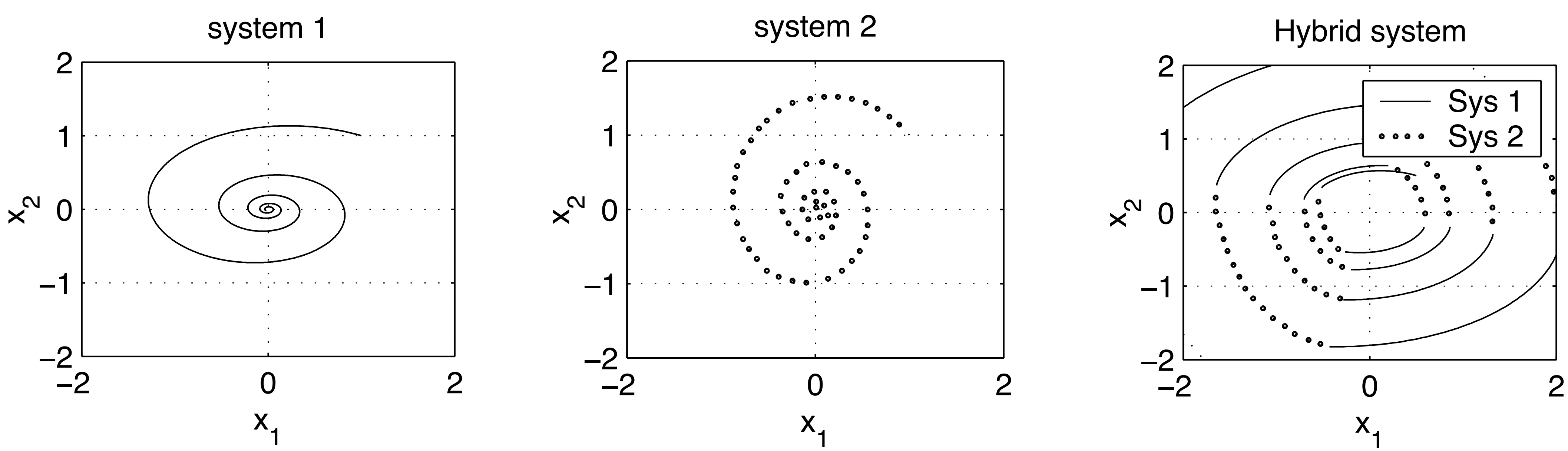
Lecture 8

Midterm Review



Stable switching function



Unstable switching function

need more than intuition to make educated judgements

Review of continuous systems

- Lypschitz continuity
- Existence and uniqueness of solutions
  (T1 and T2)
- Continuity with respect to initial conditions

$\boxed{\text{extend concepts to continuous + discrete}}$

Review (Hybrid System)

$Q = \{q_1, q_2, \ldots\}$ *is a set of* **discrete states**;

$X = \mathbb{R}^n$ *is a set of* **continuous states**;

$f(\cdot, \cdot) : Q \times X \to \mathbb{R}^n$ *is a* **vector field**;

$\mathrm{I}nit \subseteq Q \times X$ *is a set of* **initial states**;

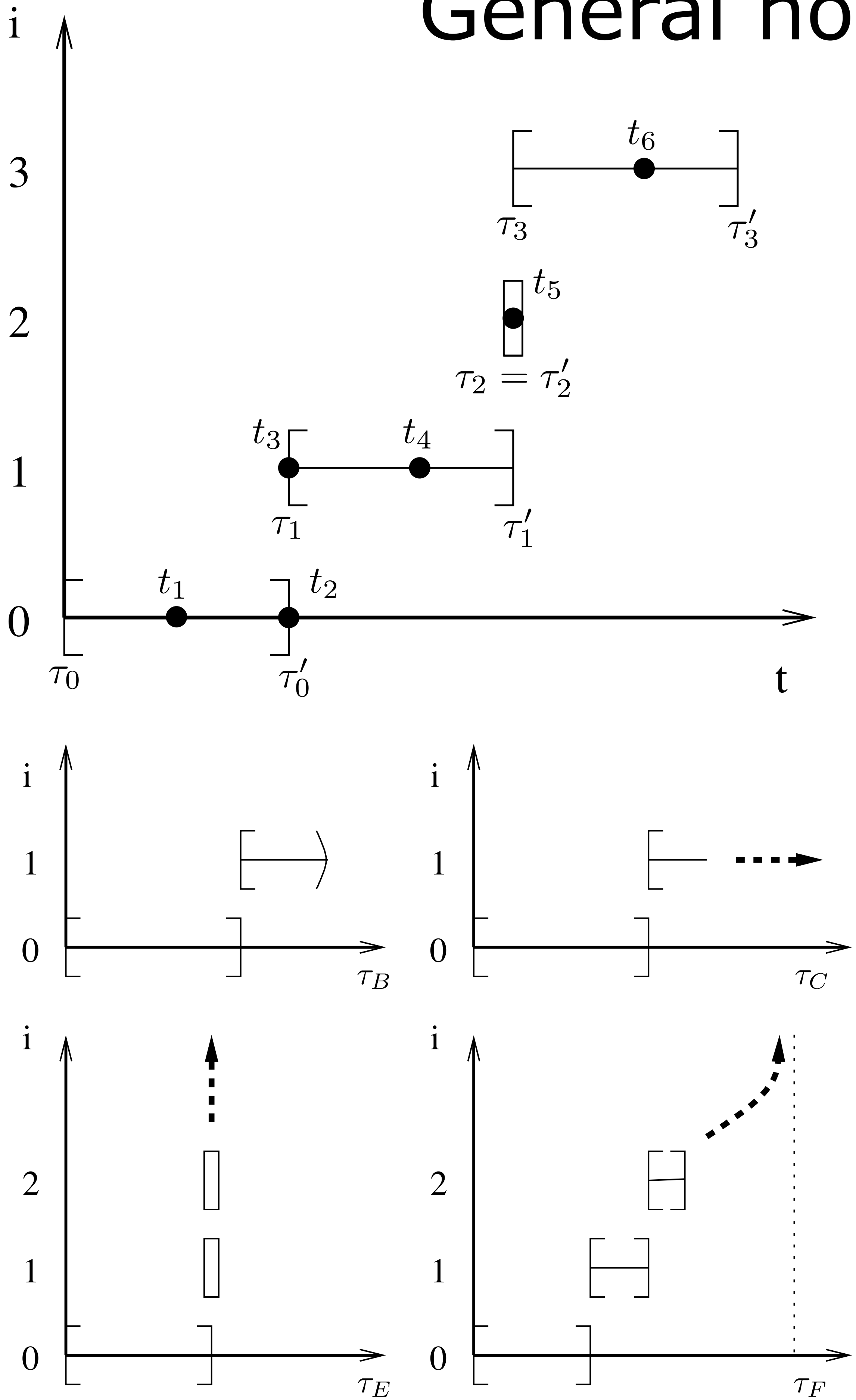
$\mathrm{D}om(\cdot) : Q \to P(X)$ *is a* **domain**;

$E \subseteq Q \times Q$ *is a set of* **edges**;

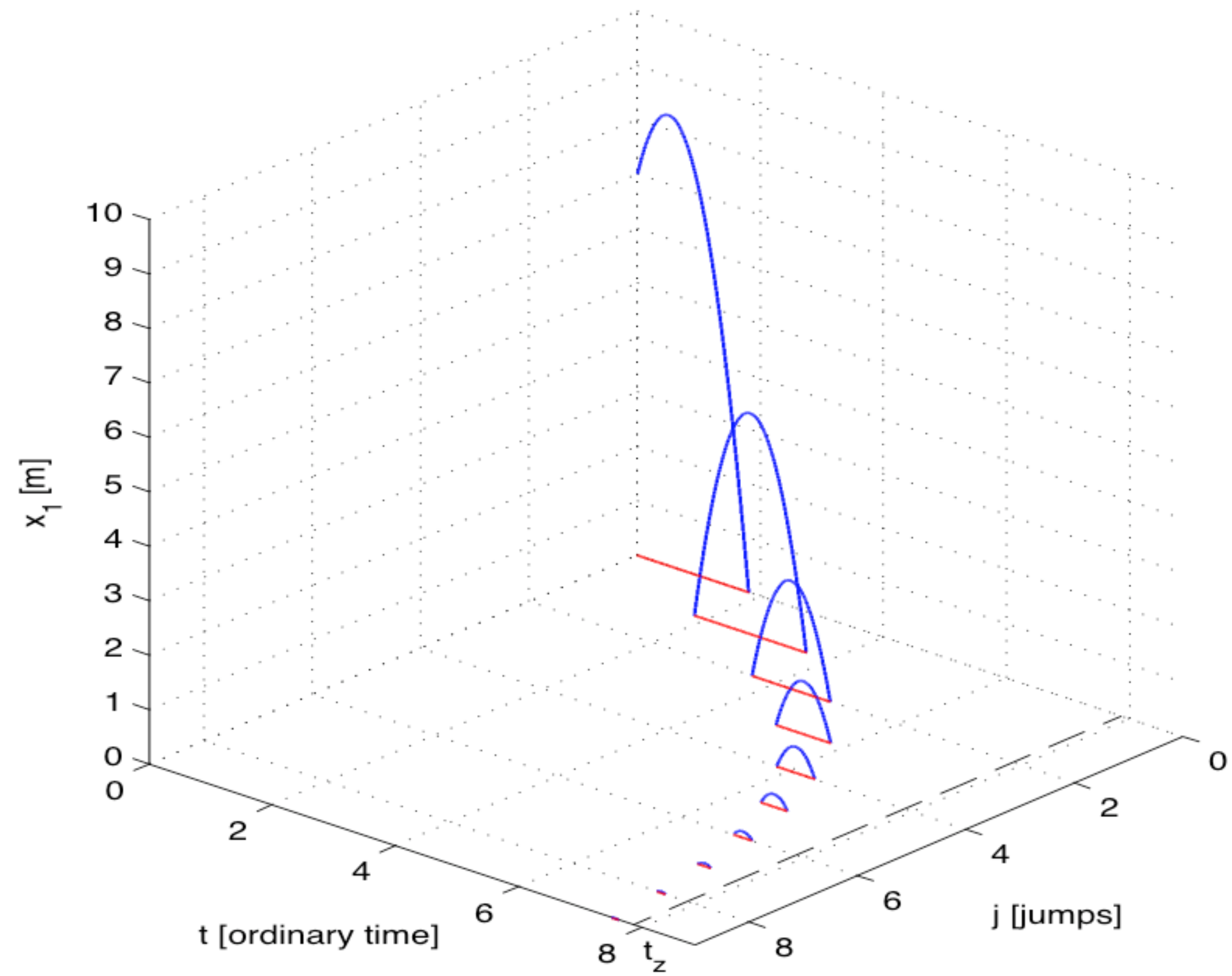
$G(\cdot) : E \to P(X)$ *is a* **guard condition**;

$R(\cdot, \cdot) : E \times X \to P(X)$ *is a* **reset map**.

General notation for HS



Generalize time + space



Generalize solutions

What can go wrong?

- Problems in the continuous evolution
  - existence
  - uniqueness
  - finite escape
- Problems in the hybrid execution:
  - chattering
  - zeno
- Non-continuous dependency on initial conditions

Consequence of over-simplified or ill-posed model!

Solution requires new concepts

- reachable states
- transition states
- deterministic v. non-deterministic
- blocking v. non-blocking

Fundamental concepts

Transition states

Set of states from which continuous evolution is impossible:

States for which continuous evolution forces the system to exit the domain instantaneously + states outside the domain

$$\mathrm{T}rans = \{(\hat{q},\hat{x}) \in Q \times X \mid \forall \epsilon > 0 \; \exists t \in [0,\epsilon) \text{ such that } (\hat{q}, x(t)) \notin \mathrm{D}om(\hat{q})\}$$

$$\bigcup_{q \in Q} \{q\} \times \mathrm{D}om(q)^c \subseteq \mathrm{T}rans$$

Fundamental concepts

Reachable states

Solutions visit certain sets

$$\mathrm{I}nit \subseteq \mathrm{R}each$$

Properties of Hybrid Systems

Non-blocking: For all reachable states for which continuous evolution is impossible, a discrete transition is possible

$$\text{C1:} \quad \forall (\hat{q}, \hat{x}) \in Reach \cap Trans \; \exists \hat{q}' \in Q : \; (\hat{q}, \hat{q}') \in E \; \wedge \; \hat{x} \in G(\hat{q}, \hat{q}')$$

Deterministic: Non-blocking ⇔ C1 holds

Lemma 1

Properties of Hybrid Systems

Deterministic:

- Discrete transitions have unique destinations
- Whenever a discrete transition is possible continuous evolution is not

$$\text{C1:} \quad \forall(\hat{q},\hat{x}) \in Reach \cap Trans \; \exists \hat{q}' \in Q : \; (\hat{q},\hat{q}') \in E \; \wedge \; \hat{x} \in G(\hat{q},\hat{q}')$$

$$\text{C2:} \quad \hat{x} \in G(\hat{q},\hat{q}') \text{ for some } (\hat{q},\hat{q}') \in E \rightarrow (\hat{q},\hat{x}) \in Trans$$

$$\text{C3:} \quad (\hat{q},\hat{q}') \in E \wedge (\hat{q},\hat{q}'') \in E \text{ with } \hat{q}' \neq \hat{q}'' \rightarrow \hat{x} \notin G(\hat{q},\hat{q}') \cap G(\hat{q},\hat{q}'')$$

$$\text{C4:} \quad (\hat{q},\hat{q}') \in E \wedge x \in G(\hat{q},\hat{q}') \rightarrow |R(q,q',x)| \leq 1$$

Lemma 2

A hybrid system has a unique infinite solution if Conditions C1-C4 are satisfied

Theorem 4

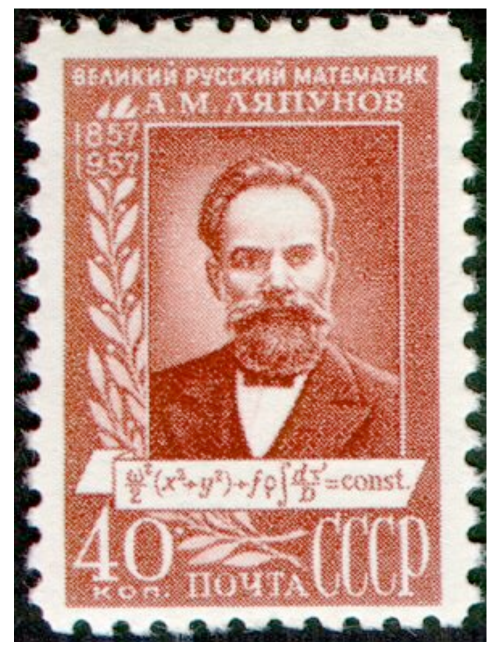
Stability

Stability for continuous systems

Aleksandr Mikhailovich Lyapunov

Born	June 6, 1857 Yaroslavl, Imperial Russia
Died	November 3, 1918 (aged 61)
Residence	Russia
Nationality	Russian
Fields	Applied mathematics
Institutions	Saint Petersburg State University Russian Academy of Sciences Kharkov University

In 1892 he was awarded the degree of Professor of Science (the second after PH.D academic degree of Russia) after defending the thesis *A general task about the stability of motion* (*Общая задача об устойчивости движения*). The theory of stability has been studied extensively by many researchers.



Issued by the Soviet Union on June 20, 1957, on the 100th anniversary of his birth

Stability for switched systems

- Multiple Lyapunov functions
- Common Lyapunov function
- Design stabilizing switching sequence

Stability for hybrid systems

Theorem 5
Consider a hybrid automaton H with $|Q| < \infty$. Consider an open set $D \subseteq X$ with $0 \in D$ and a function $V : Q \times D \to \mathbb{R}$ continuous in x with $V(q, 0) = 0$ and $V(q, x) > 0$ for all $x \in D - \{0\}$. Assume that for all (τ, q, x) the sequence $\{V(q(\tau_i), x(\tau_i)\}$ is non-increasing and that there exists a continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0)$ such that for all $t \in [\tau_i, \tau_i']$, $V(q(t), x(t)) \leq g(V(q(\tau_i), x(\tau_i))$ then $x = 0$ is a stable equilibrium of H.

Example

$\mathbf{Q} = \{q_1, q_2\}$ and $\mathbf{X} = \mathbb{R}^2$;

$\mathrm{Init} = \mathbf{Q} \times \{x \in \mathbf{X} : \|x > 0\|\}$;

$f(q_1, x) = A_1 x$ and $f(q_2, x) = A_2 x$, with:

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$

$I(q_1) = \{x \in \mathbf{X} : Cx \geq 0\}$ and $I(q_2) = \{x \in \mathbf{X} : Cx \leq 0\}$;

$E = \{(q_1, q_2), (q_2, q_1)\}$;

$G(q_1, q_2) = \{x \in \mathbf{X} : Cx \leq 0\}$ and $G(q_2, q_1) = \{x \in \mathbf{X} : Cx \geq 0\}$; and

$R(q_1, q_2, x) = R(q_2, q_1, x) = \{x\}$.

show that the origin is a stable equilibrium point

Example

Step 1

$f(q_1, 0) = f(q_1, 0) = 0$ and $R(q, q', 0) = \{0\}$

Step 2

If $\dot{x} = A_i x$ *is stable, then there exists* $P_i = P_i^T > 0$ *such that* $A_i^T P_i + P_i A_i = -I$

Step 3

Recall that $P_i > 0$ (positive definite) if and only if $x^T P_i x > 0$ for all $x \neq 0$, and I is the identity matrix.
Consider the candidate Lyapunov function:

$$V(q,x) = \begin{cases} x^T P_1 x & \text{if } q = q_1 \\ x^T P_2 x & \text{if } q = q_2 \end{cases}$$

P1 =

0.2752	-0.0225
-0.0225	2.7478

P2 =

2.7478	0.0225
0.0225	0.2752

Example

Step 4

Check that the conditions of the Theorem hold. For all $q \in \mathbf{Q}$:

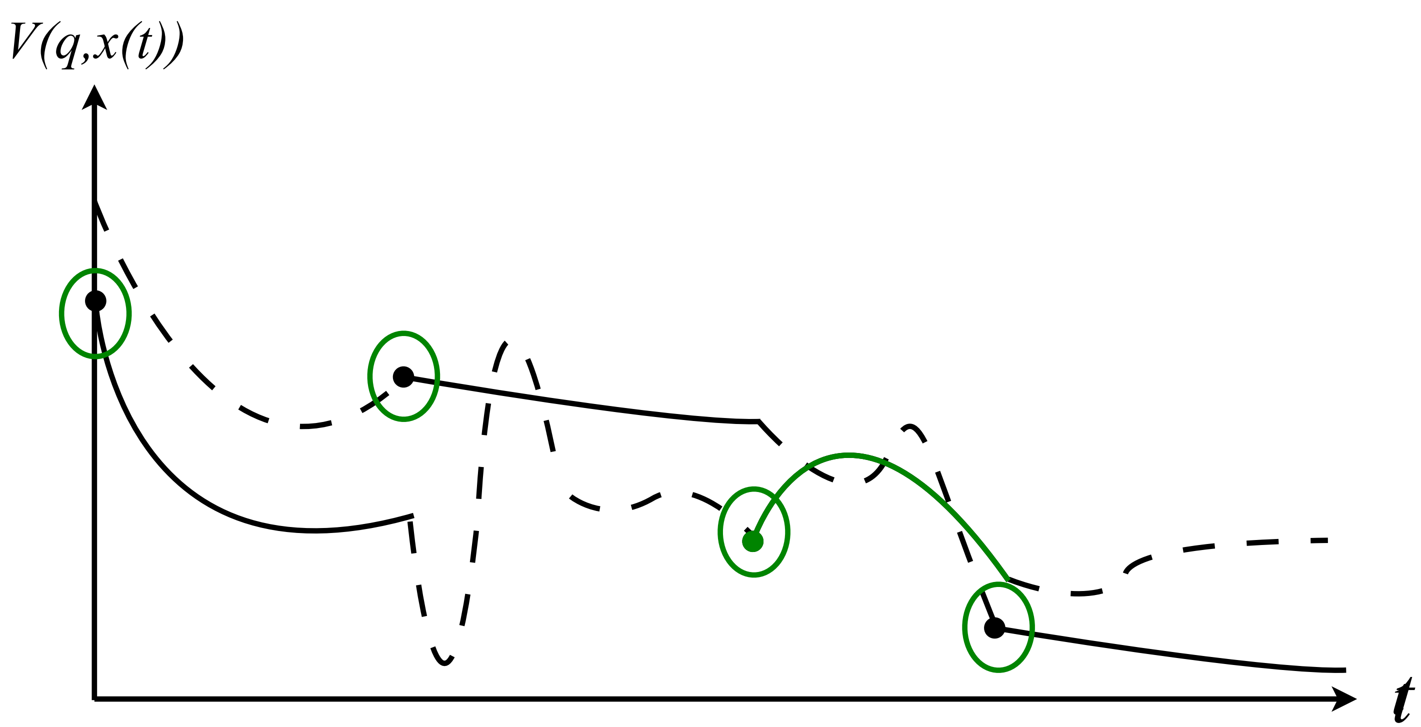
1. $V(q, 0) = 0$
2. $V(q, x) > 0$ for all $x \neq 0$ (since the P_i are positive definite)
3. $\frac{\partial V}{\partial x}(q, x) f(q, x) \leq 0$ for all x since:

$$\begin{aligned}
\frac{\partial V}{\partial x}(q, x) f(q, x) &= \frac{d}{dt} V(q, x(t)) \\
&= \dot{x}^T P_i x + x^T P_i \dot{x} \\
&= x^T A_i^T P_i x + x^T P_i A_i x \\
&= x^T (A_i^T P_i + P_i A_i) x \\
&= -x^T I x
\end{aligned}$$

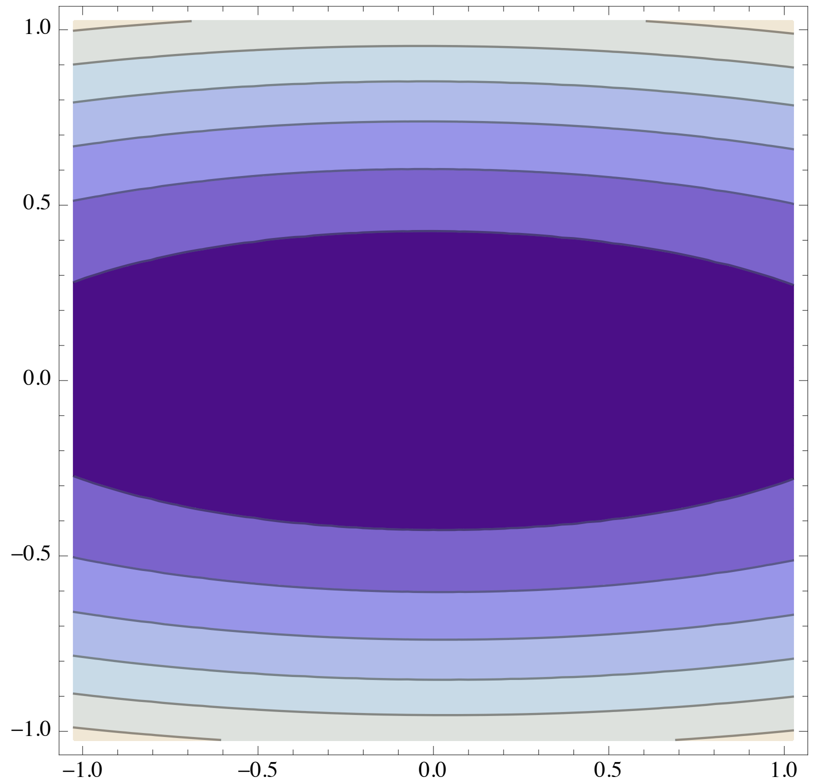
Example

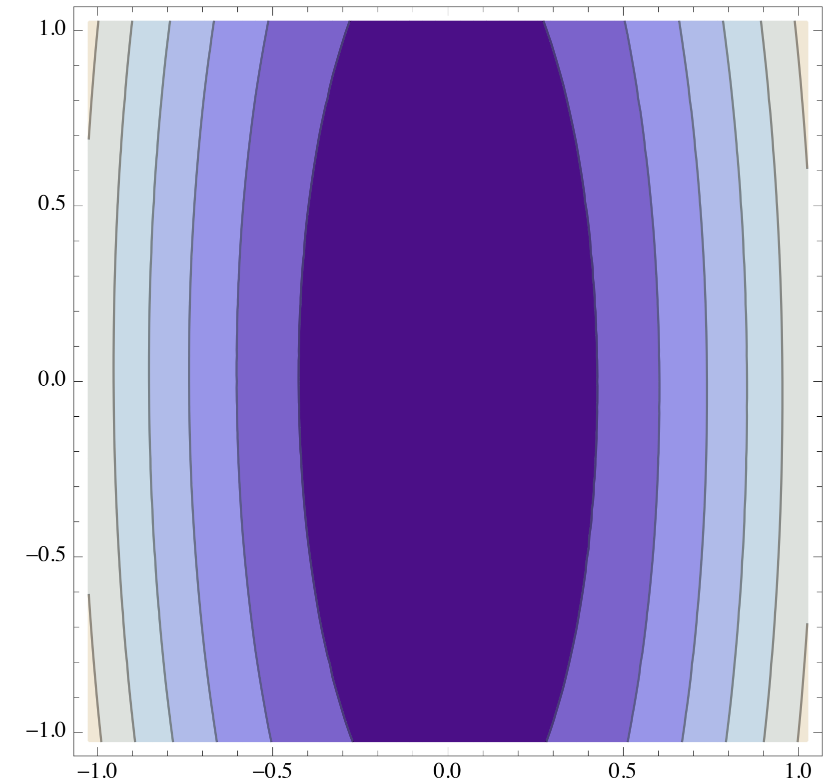
Step 5

It remains to test the non-increasing sequence condition. Notice that the level sets of $x^T P_i x$ are ellipses centered at the origin. Therefore each level set intersects the switching line $Cx = 0$ at exactly two points, $\hat{x}$ and $-\hat{x}$. Assume $x(\tau_i) = \hat{x}$ and $q(\tau_i) = q_1$. The fact that $V(q_1, x(t))$ decreases for $t \in [\tau_i, \tau_i']$ (where $q(t) = q_1$) implies that the next time the switching line is reached, $x(\tau_i') = \alpha(-\hat{x})$ for some $\alpha \in (0,1)$. Therefore, $\|x(\tau_{i+1})\| = \|x(\tau_i')\| < \|x(\tau_i)\|$. By a similar argument, $\|x(\tau_{i+2})\| = \|x(\tau_{i+1}')\| = \|x(\tau_{i+1})\|$ and $Cx(\tau_{i+2}) = 0$. Therefore, $V(q(\tau_i), x(\tau_i)) \leq V(q(\tau_{i+2}), x(\tau_{i+2}))$.

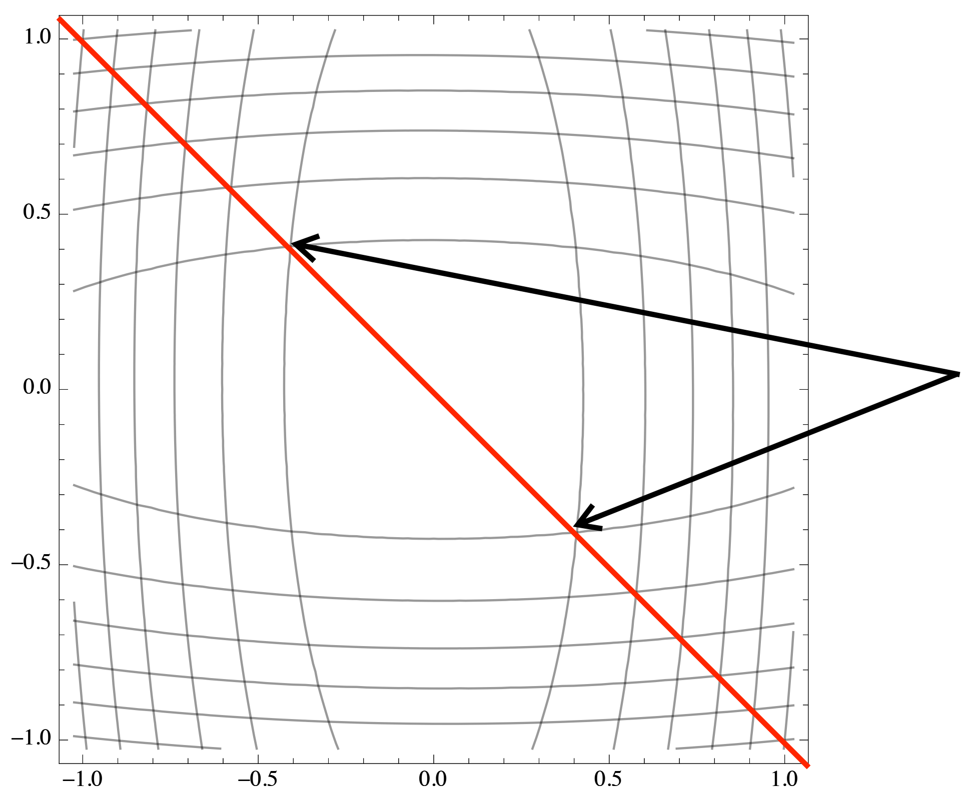


Example





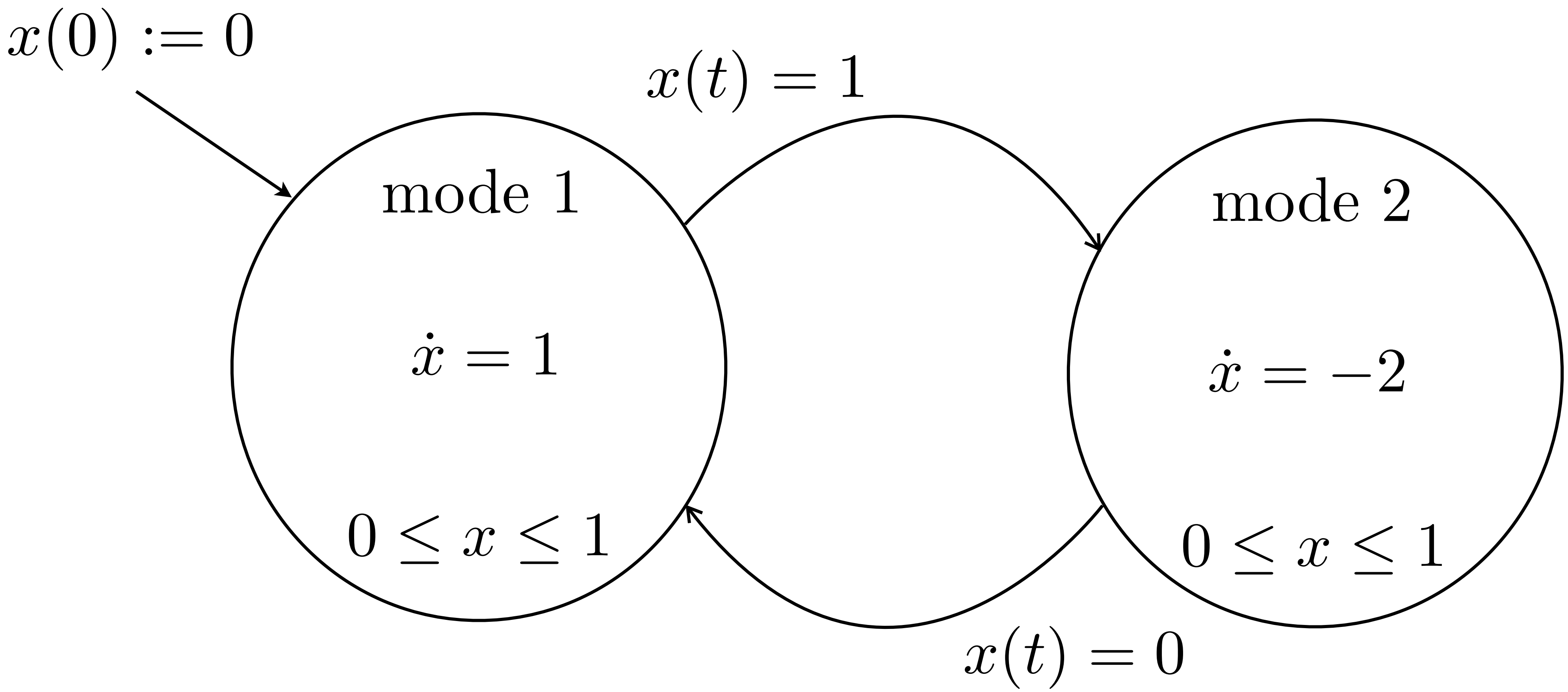
$C = [1 \;\; 1]$



level set intersect switching line only at two points

Exercises

For the following hybrid system:



Derive the hybrid model $H = \{Q, X, f, Int, D, E, G, R\}$
Is H non-blocking?
Is H deterministic?
Sketch the state trajectory for both the mode and the continuous state