Lecture 16

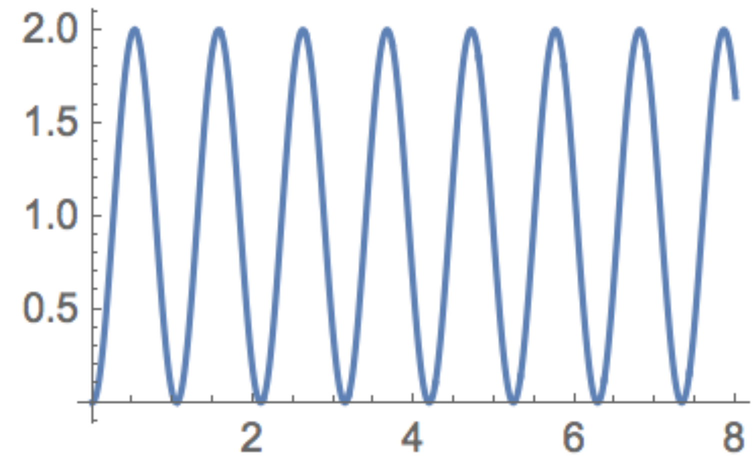
- why the frequency domain? -

Undamped

$\zeta = 0$

$$\left(\frac{36}{36+s^2} \right)$$

s_1 = -6i, s_2 =6i

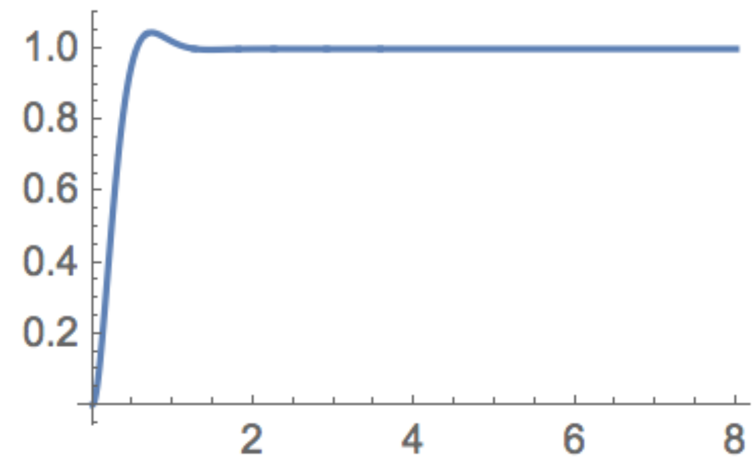


Underdamped

$0 < \zeta < 1$

$$\left(\frac{36}{36+8.4\,s+s^2} \right)$$

s_1 = -4.2 - 4.2i, s_2 = -4.2 + 4.2i

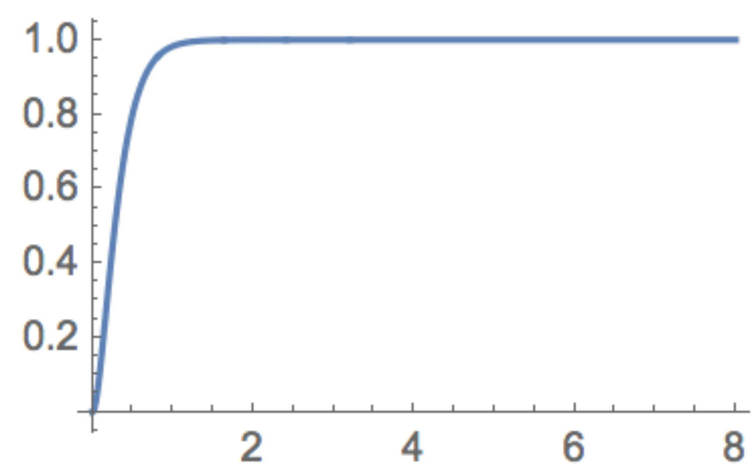


Critically damped

$\zeta = 1$

$$\left(\frac{36}{36+12\,s+s^2} \right)$$

s_1 = -6, s_2 = -6

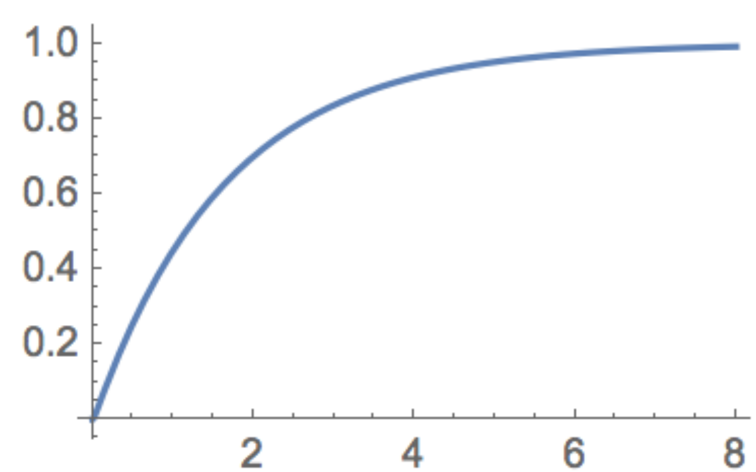


Overdamped

$\zeta > 1$

$$\left(\frac{36}{36+60\,s+s^2} \right)$$

s_1 = -59, s_2 = -0.6

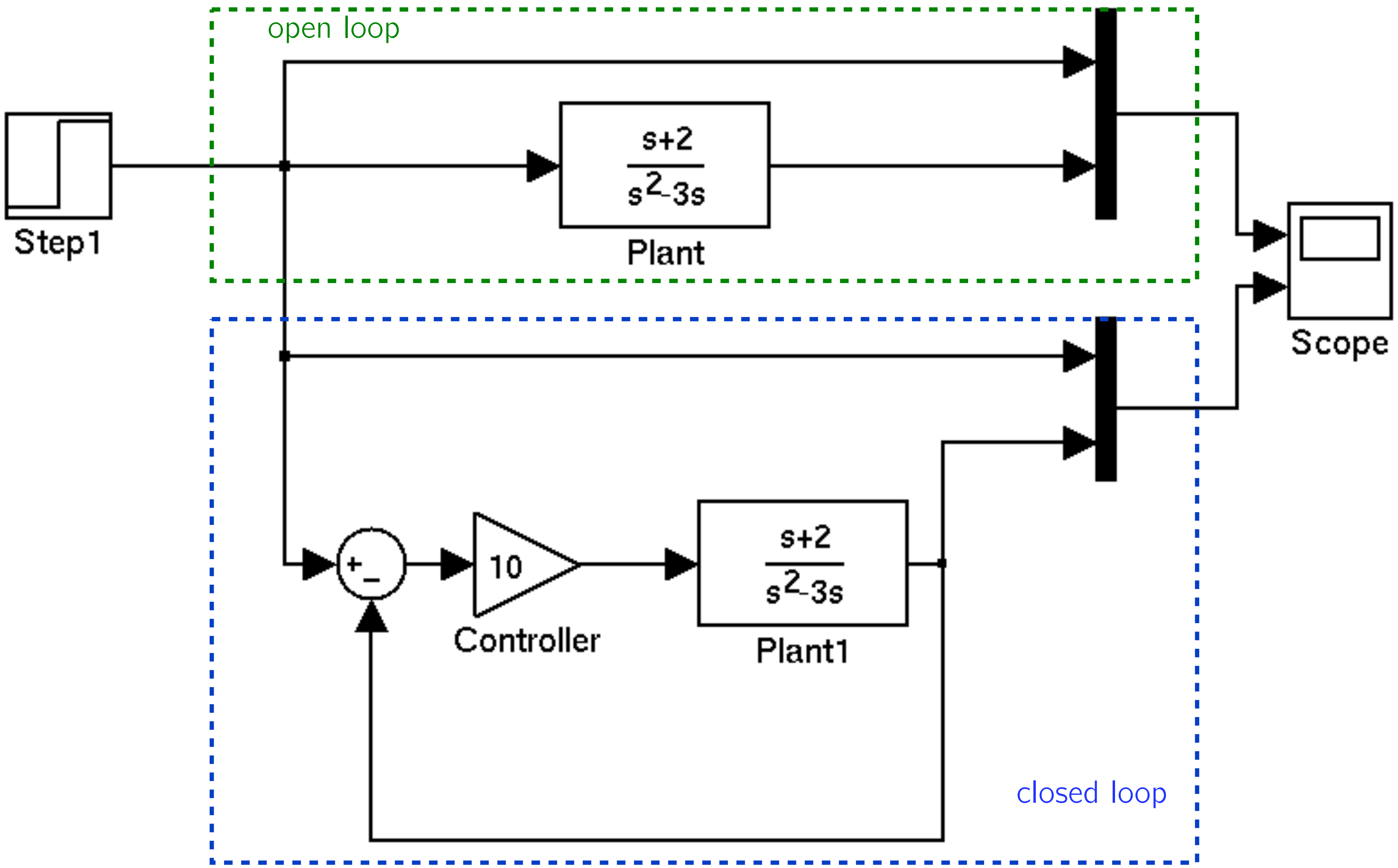


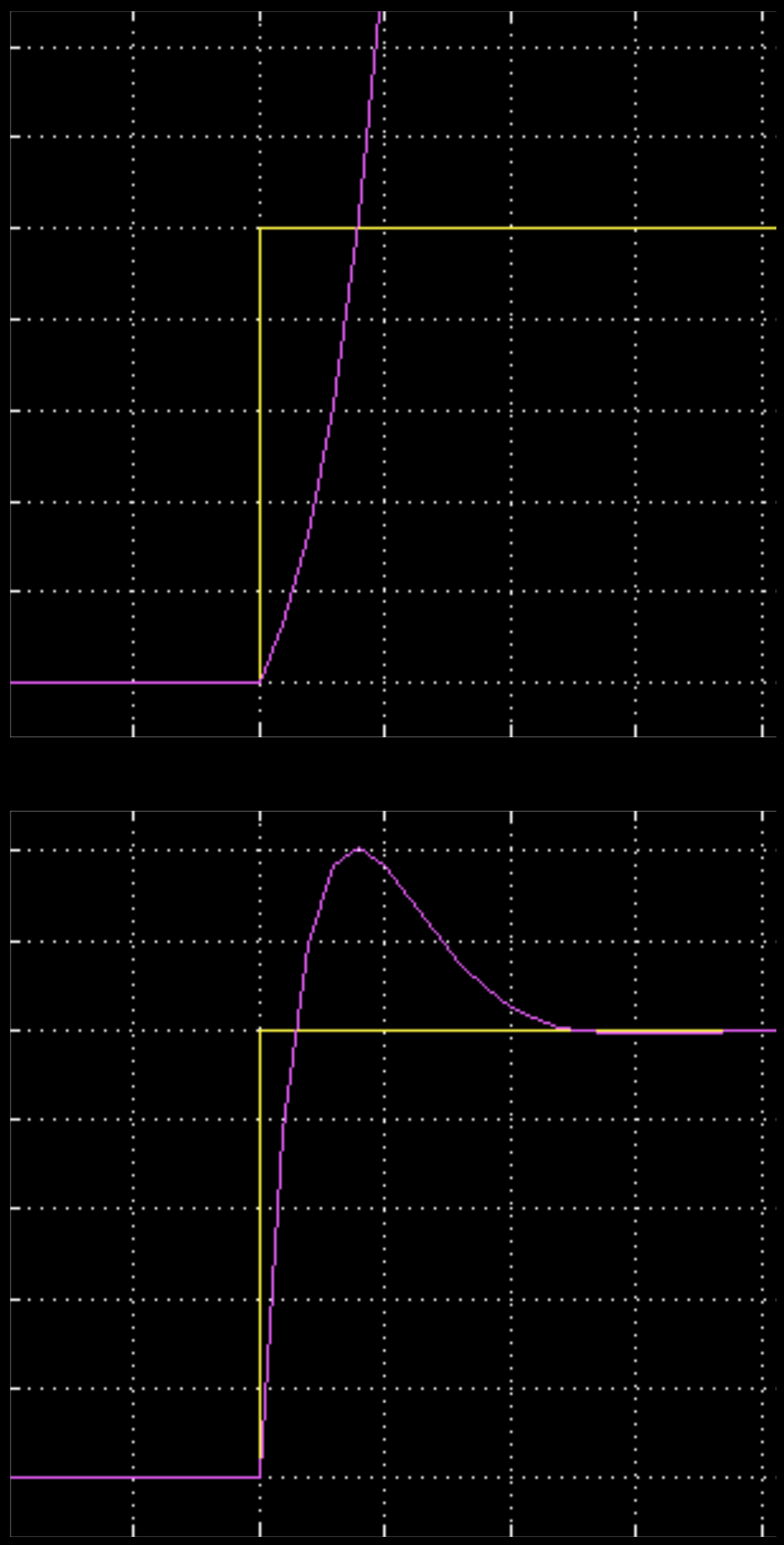
Review

- Transfer functions
- Realization of transfer functions

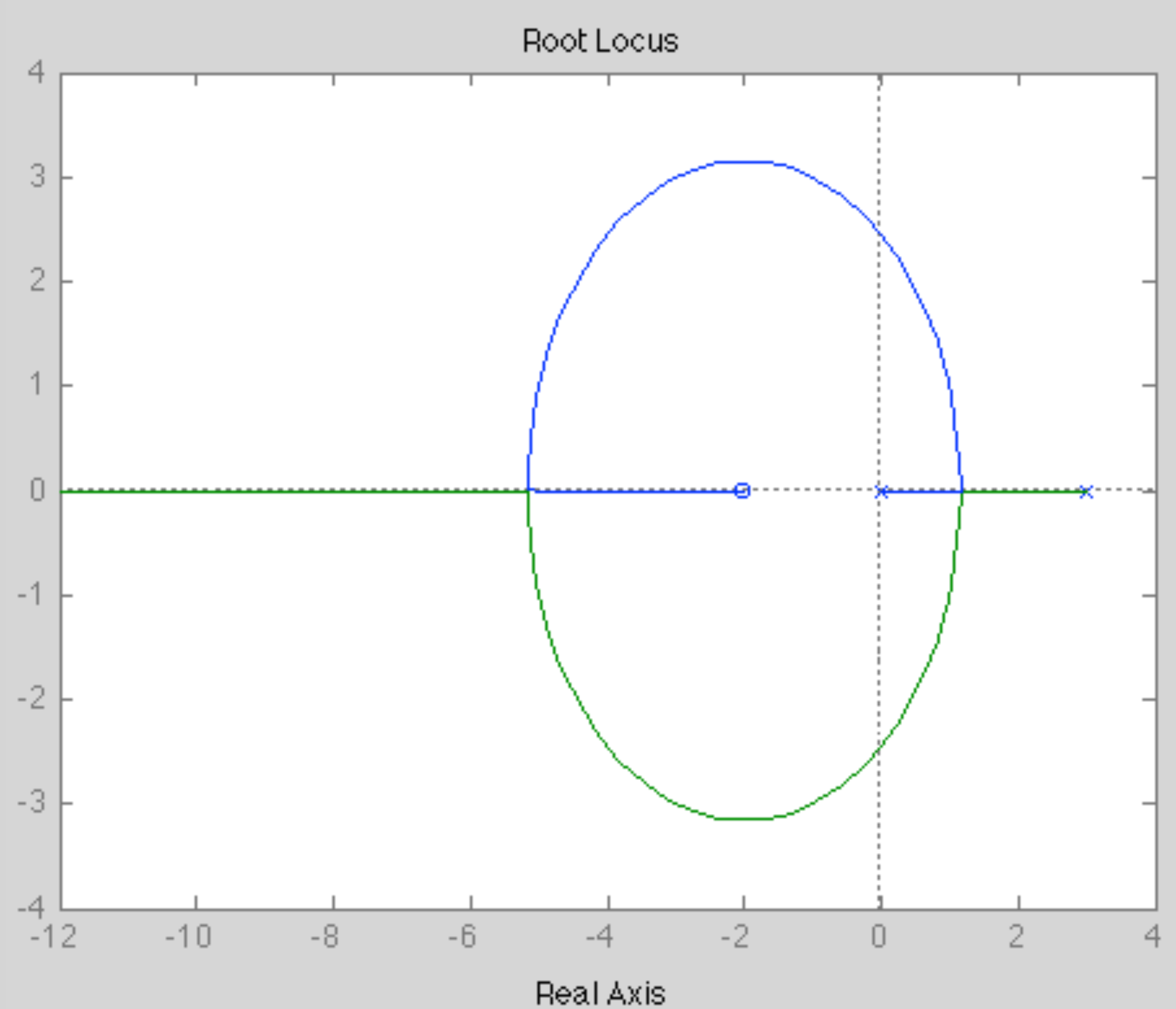
Today

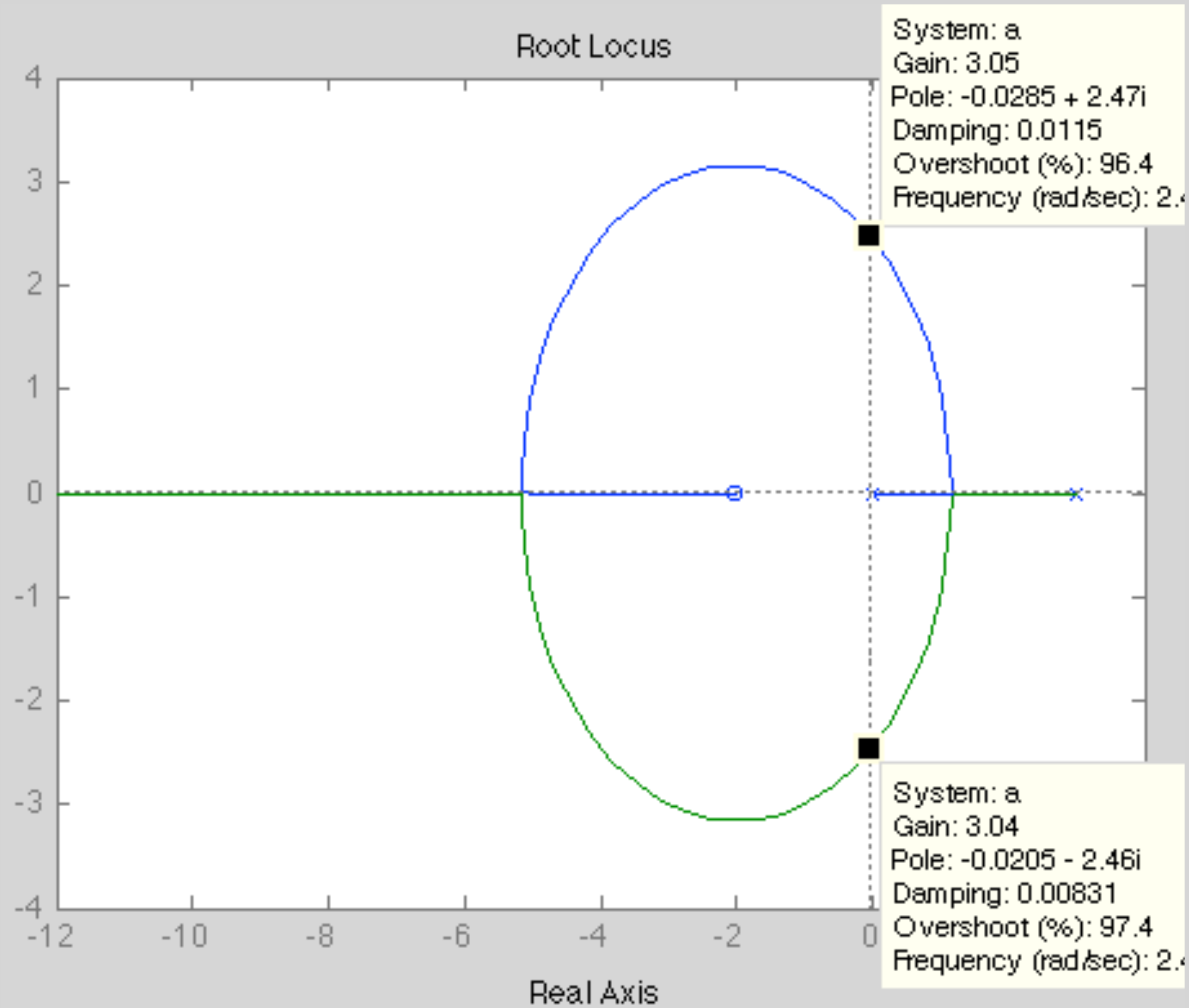
- Root locus
- Frequency response (Bode plot)
- Block diagram algebra for interconnected systems

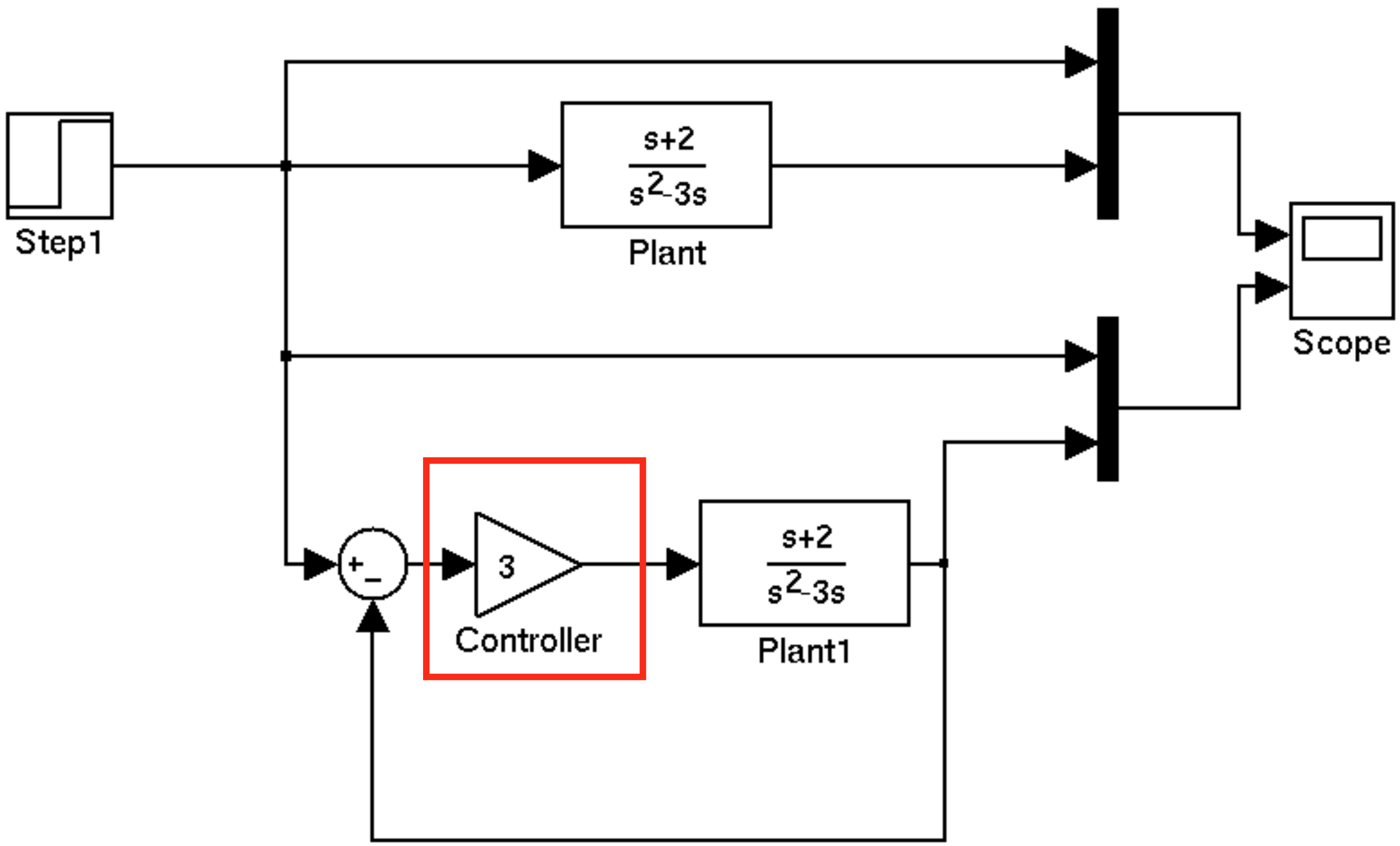

open loop
Step1
s+2
s2-3s
Plant
Scope
+
−
10
Controller
s+2
s2-3s
Plant1
closed loop



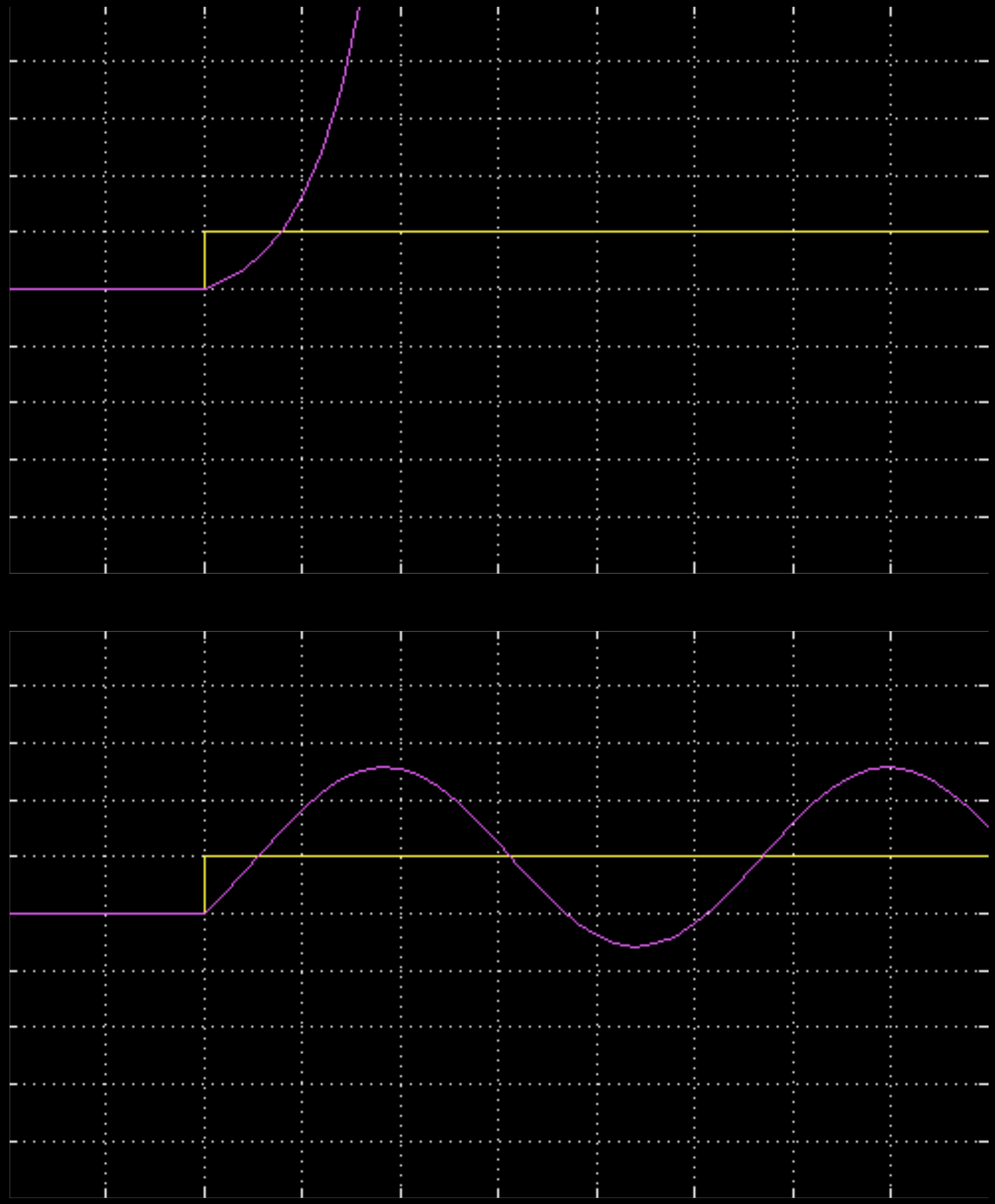
Feedback can stabilize an equilibrium point


Root Locus
4
3
2
1
0
-1
-2
-3
-4
-12
-10
-8
-6
-4
-2
0
2
4
Real Axis

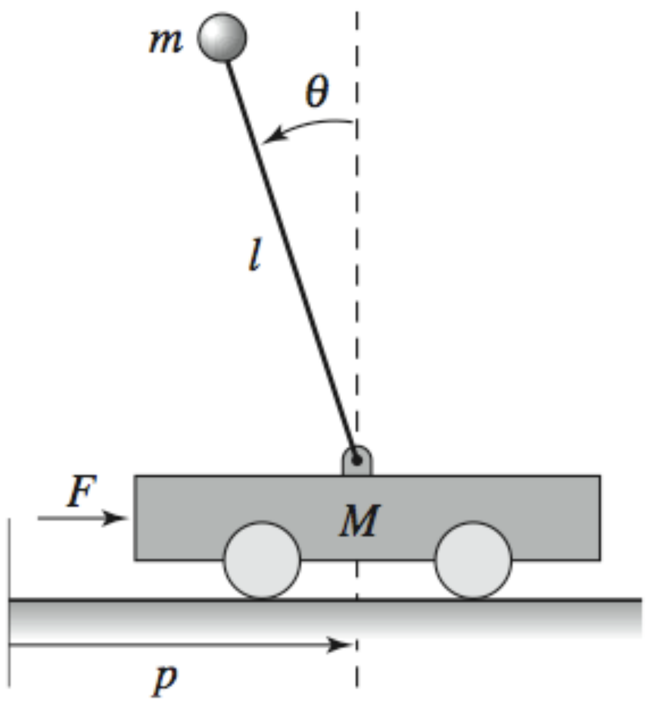

Root Locus
System: a
Gain: 3.05
Pole: -0.0285 + 2.47i
Damping: 0.0115
Overshoot (%): 96.4
Frequency (rad/sec): 2.
System: a
Gain: 3.04
Pole: -0.0205 - 2.46i
Damping: 0.00831
Overshoot (%): 97.4
Frequency (rad/sec): 2.
4
3
2
1
0
-1
-2
-3
-4
-12
-10
-8
-6
-4
-2
0
Real Axis



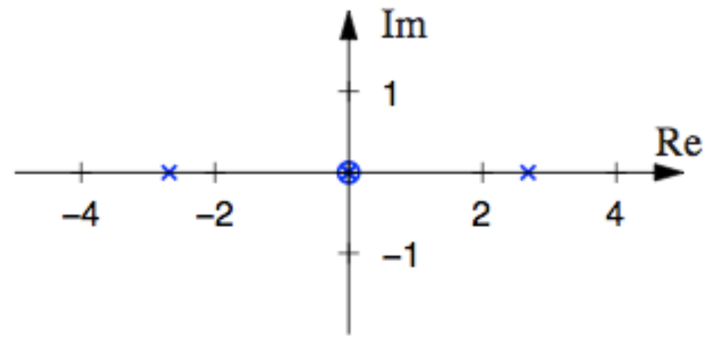
s+2
s2-3s
Step1
Plant
Scope
3
Controller
s+2
s2-3s
Plant1



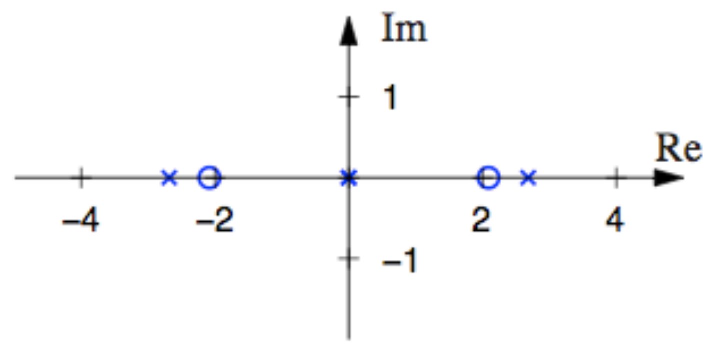
Zero-Pole Diagram



(a) Cart–pendulum system



(b) Pole zero diagram for $H_{\theta F}$



(c) Pole zero diagram for H_{pF}

pole is marked with a cross, and each zero with a circle

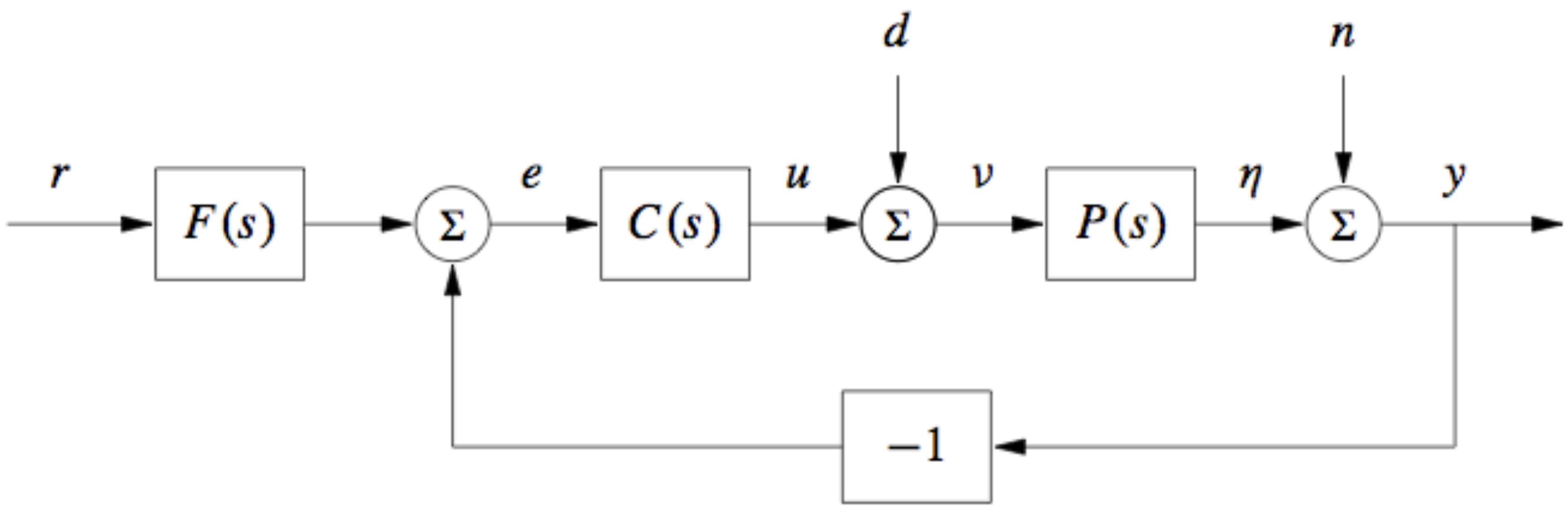
$$H_{\theta F} = \frac{mls}{(M_t J_t - m^2 l^2)s^4 + (\gamma M_t + cJ_t)s^2 + (c\gamma - M_t mgl)s - mglc},$$

$$H_{pF} = \frac{J_t s^2 + \gamma s - mgl}{(M_t J_t - m^2 l^2)s^4 + (\gamma M_t + cJ_t)s^3 + (c\gamma - M_t mgl)s^2 - mglcs},$$

$c = 0$ and $\gamma = 0$,

$$p = \pm\sqrt{\frac{mgl M_t}{M_t J_t - m^2 l^2}} \approx \pm 2.68, \qquad z = \pm\sqrt{\frac{mgl}{J_t}} \approx \pm 2.09.$$

Control System

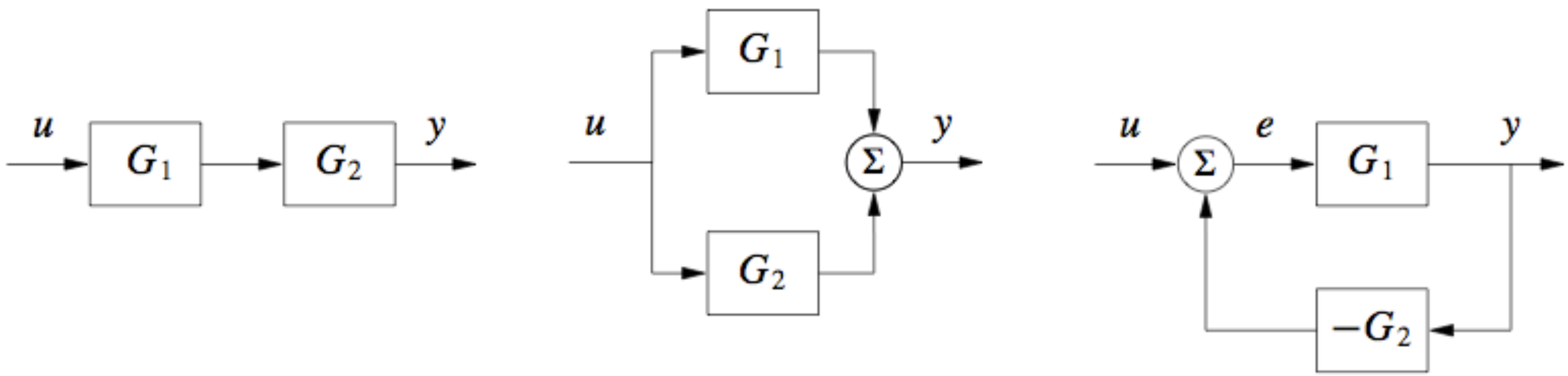


$$e = Fr - y. \qquad y = n + \eta, \qquad \eta = P(d+u), \qquad u = Ce.$$

$$e = G_{er}r + G_{en}n + G_{ed}d,$$

$$G_{yr} = \frac{PCF}{1+PC}. \qquad G_{er} = \frac{F}{1+PC}, \qquad G_{en} = \frac{-1}{1+PC}, \qquad G_{ed} = \frac{-P}{1+PC}$$

Block Diagrams and Transfer Functions

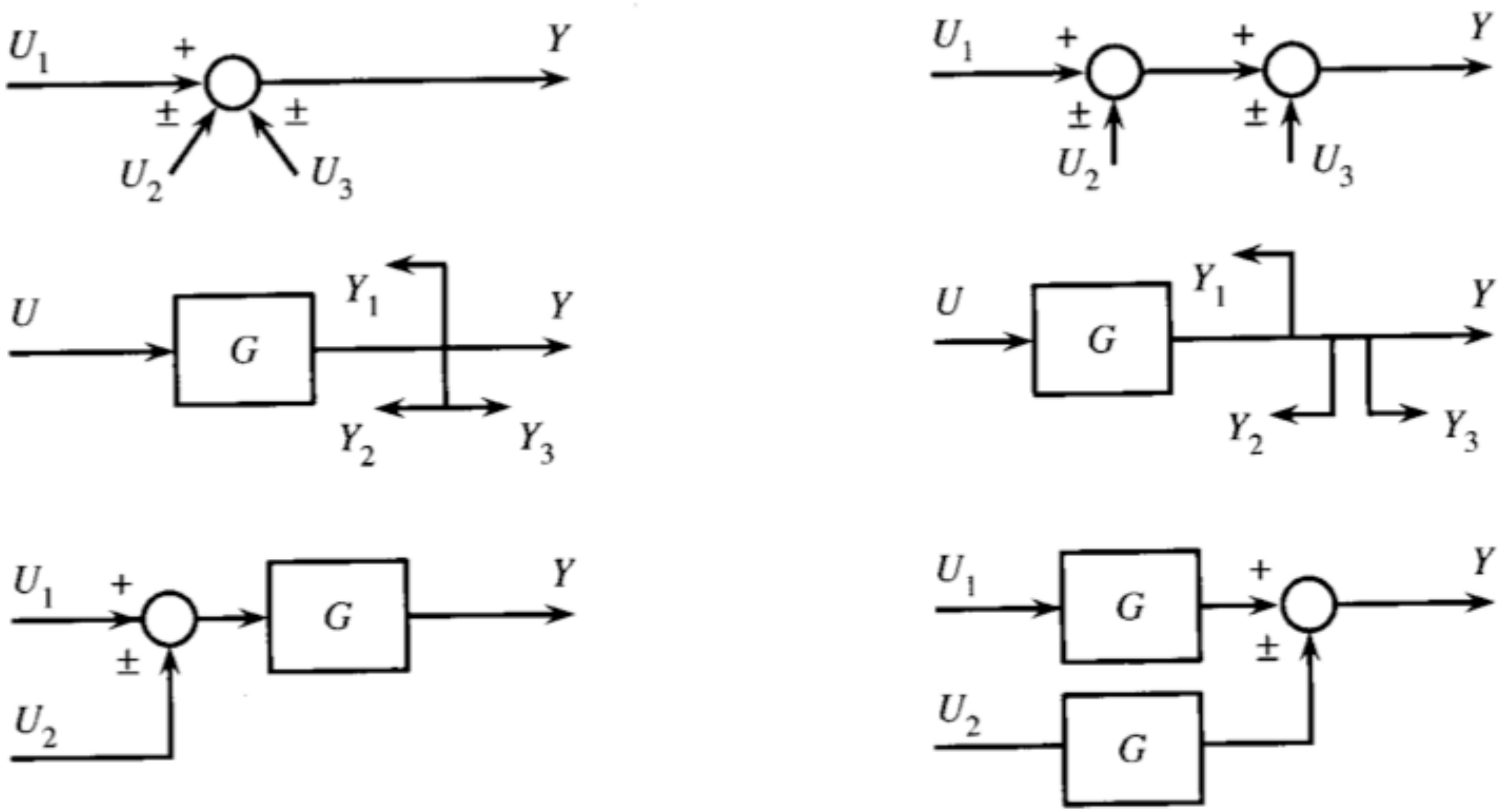


(a) $G_{yu} = G_2 G_1$

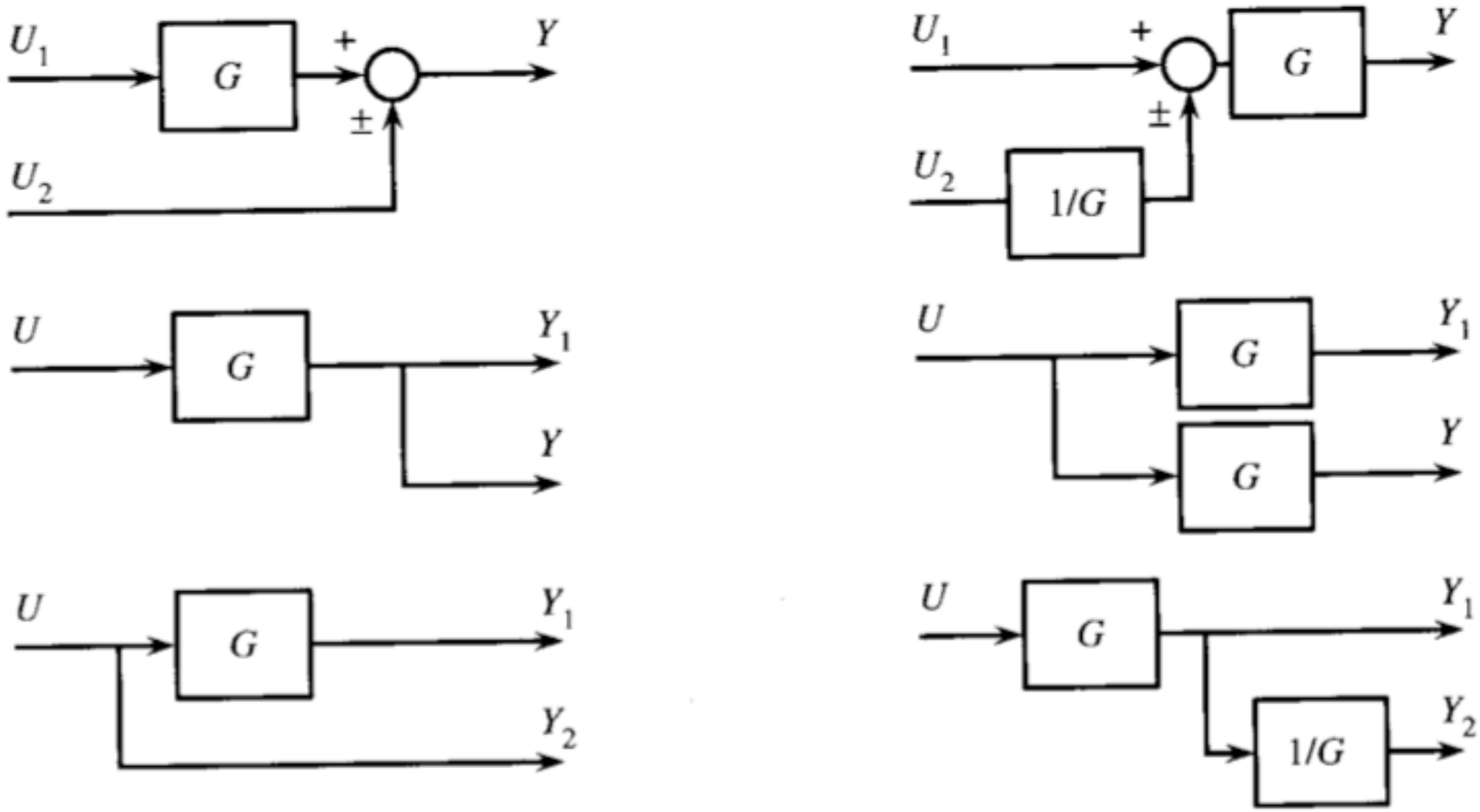
(b) $G_{yu} = G_1 + G_2$

(c) $G_{yu} = \dfrac{G_1}{1 + G_1 G_2}$

Block Algebra



Block Algebra



The Bode Plot

- Popular:easy to sketch + interpret, covers the behavior over a wide frequency range (log scale)

- The frequency response of a linear system can be computed from its transfer function by setting $s{=}i\omega$

- Th

$$u(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

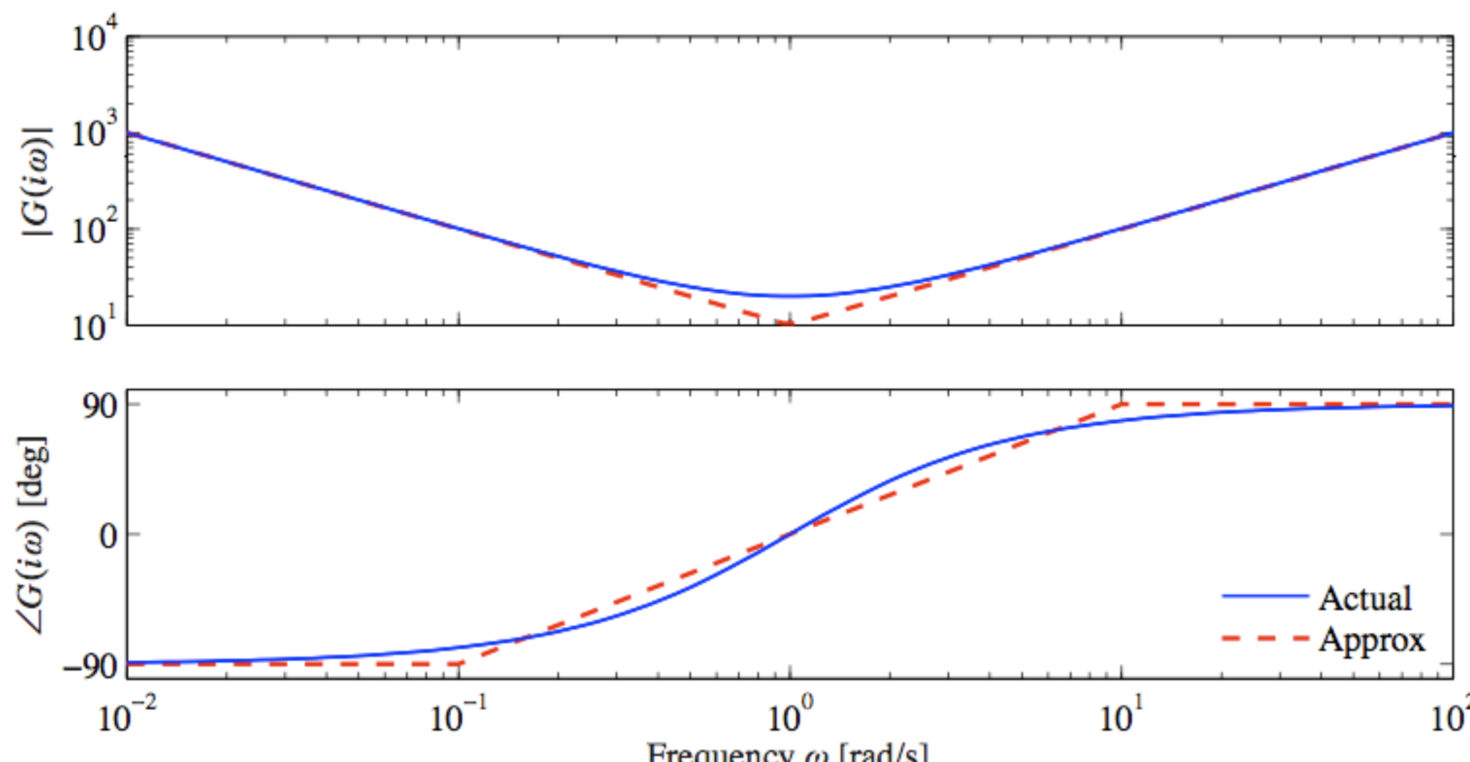
$$y(t) = G(i\omega)e^{i\omega t} = Me^{i\omega t+\varphi} = M\cos(\omega t+\varphi) + i\,M\sin(\omega t+\varphi),$$

Gain

$$M = |G(i\omega)|$$

Phase

$$\varphi = \arctan\frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}$$



Sketching Bode Plots

Consider a rational function of the form

$$G(s) = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}.$$

since

$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|$$

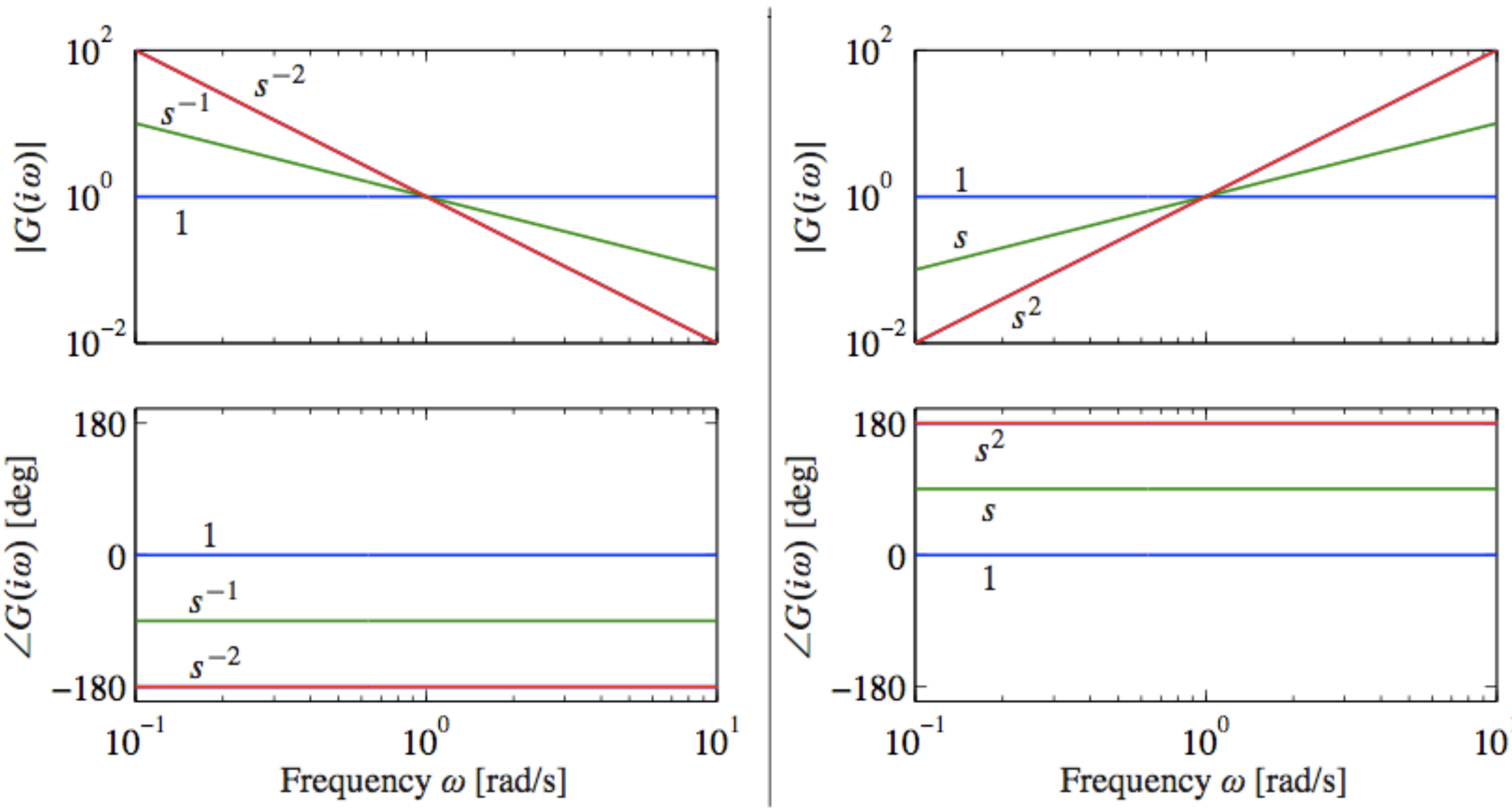
we can compute the gain curve by simply adding and subtracting gains.

Polynomials can be written as a product of terms, it suffices to know sketches for

$$\boxed{k, \quad s, \quad s+a, \quad s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$G(s) = s^k$$

$$\log|G(i\omega)| = k \log \omega, \quad \angle G(i\omega) = 90k.$$



Asymptotic Approximation

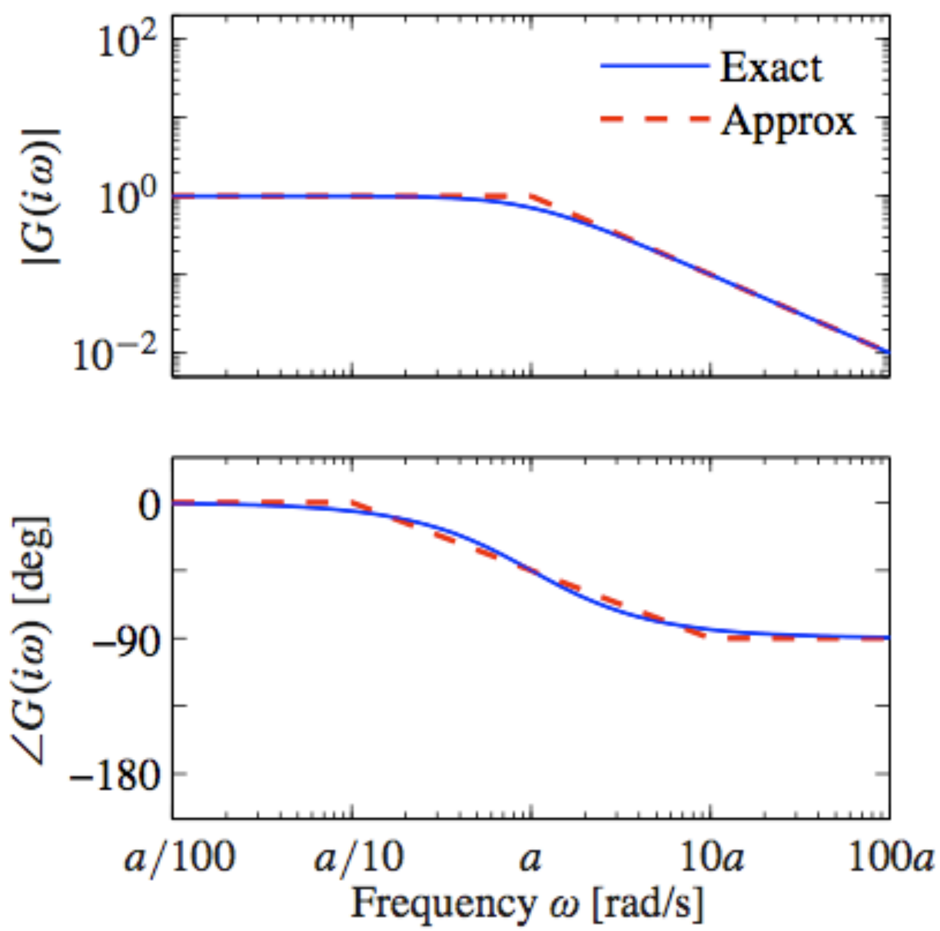
$$G(s) = \frac{a}{s+a}.$$

$$\log|G(i\omega)| = \log a - \frac{1}{2}\log(\omega^2 + a^2),$$

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{\omega}{a}.$$

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega < a \\ \log a - \log\omega & \text{if } \omega > a, \end{cases}$$

$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega < a/10 \\ -45 - 45(\log\omega - \log a) & a/10 < \omega < 10a \\ -90 & \text{if } \omega > 10a. \end{cases}$$



(a) First-order system

Asymptotic Approximation

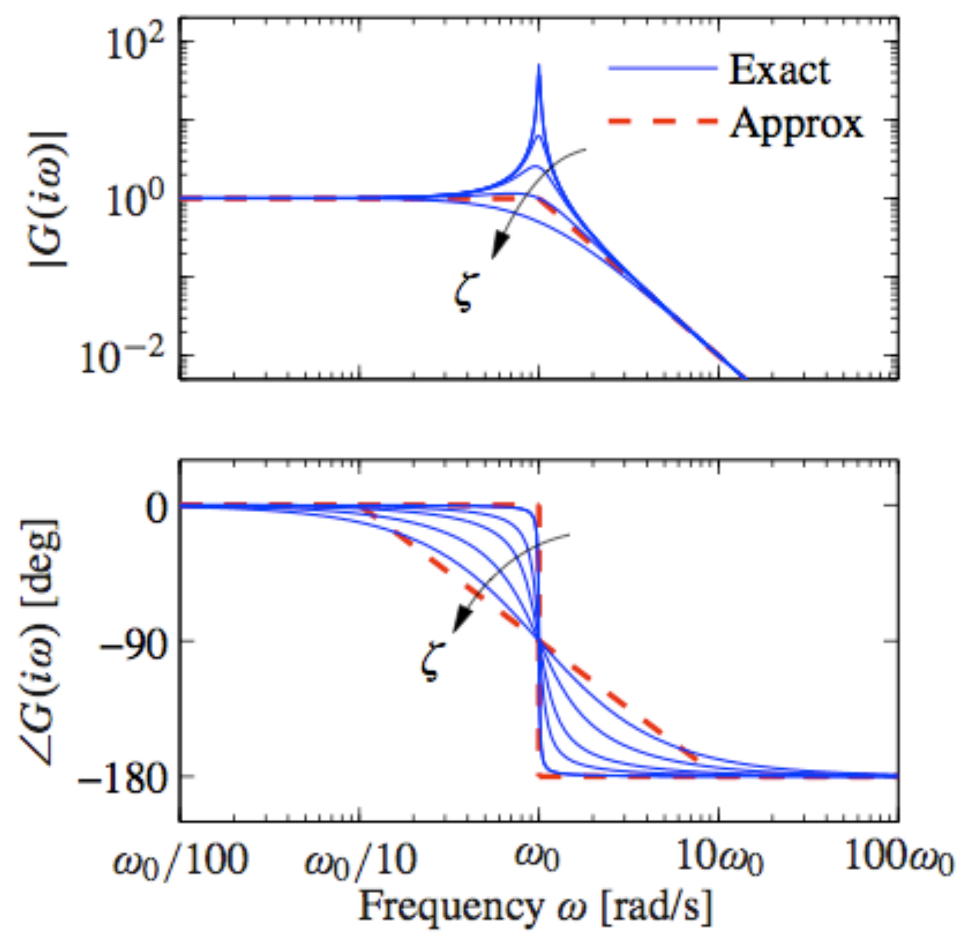
$$G(s) = \frac{\omega_0^2}{s^2 + 2a\zeta s + \omega_0^2},$$

$$\log|G(i\omega)| = 2\log\omega_0 - \frac{1}{2}\log\left(\omega^4 + 2\omega_0^2\omega^2(2\zeta^2 - 1) + \omega_0^4\right),$$

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2}.$$

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega & \text{if } \omega \gg \omega_0, \end{cases}$$

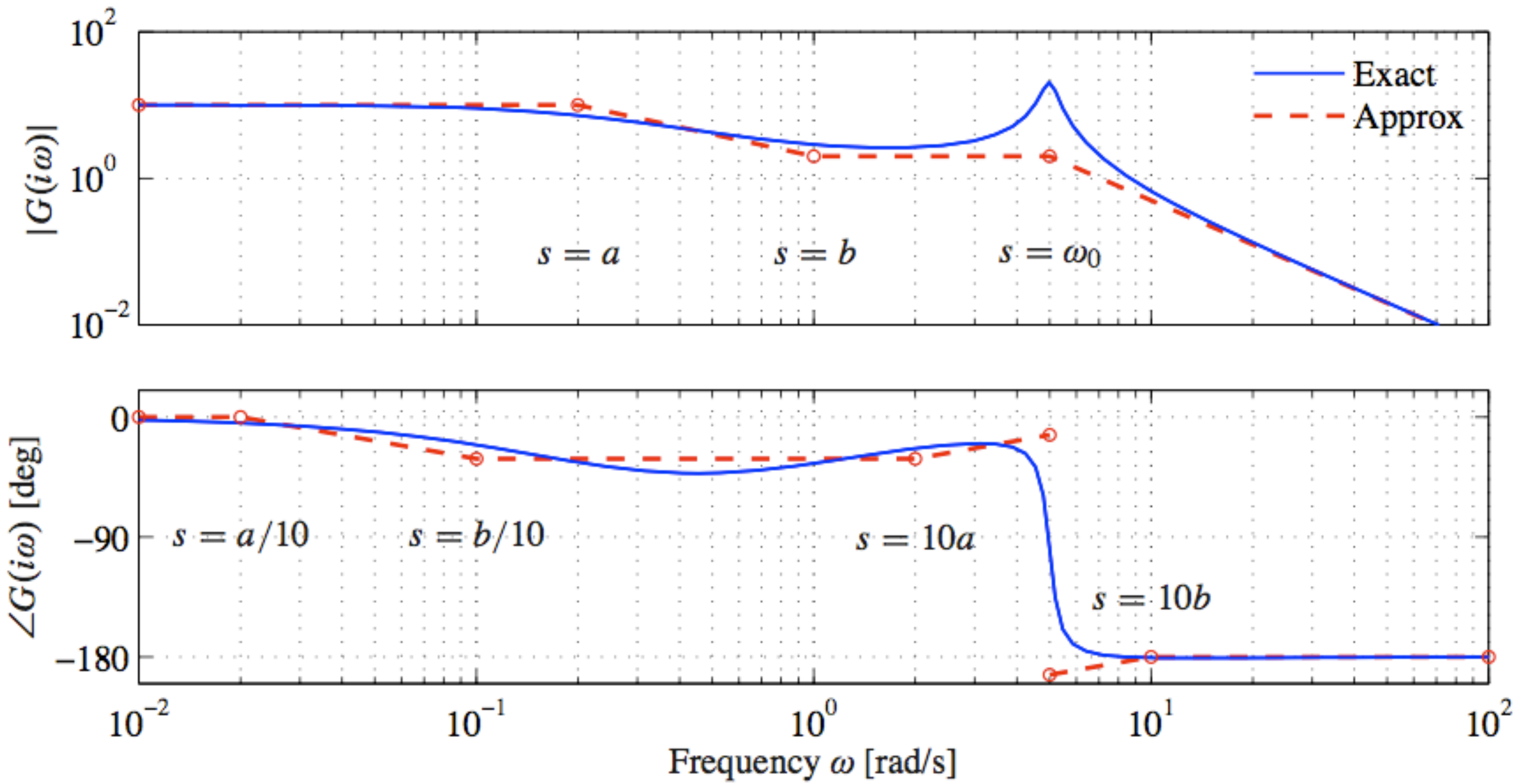
$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ -180 & \text{if } \omega \gg \omega_0. \end{cases}$$



(b) Second-order system

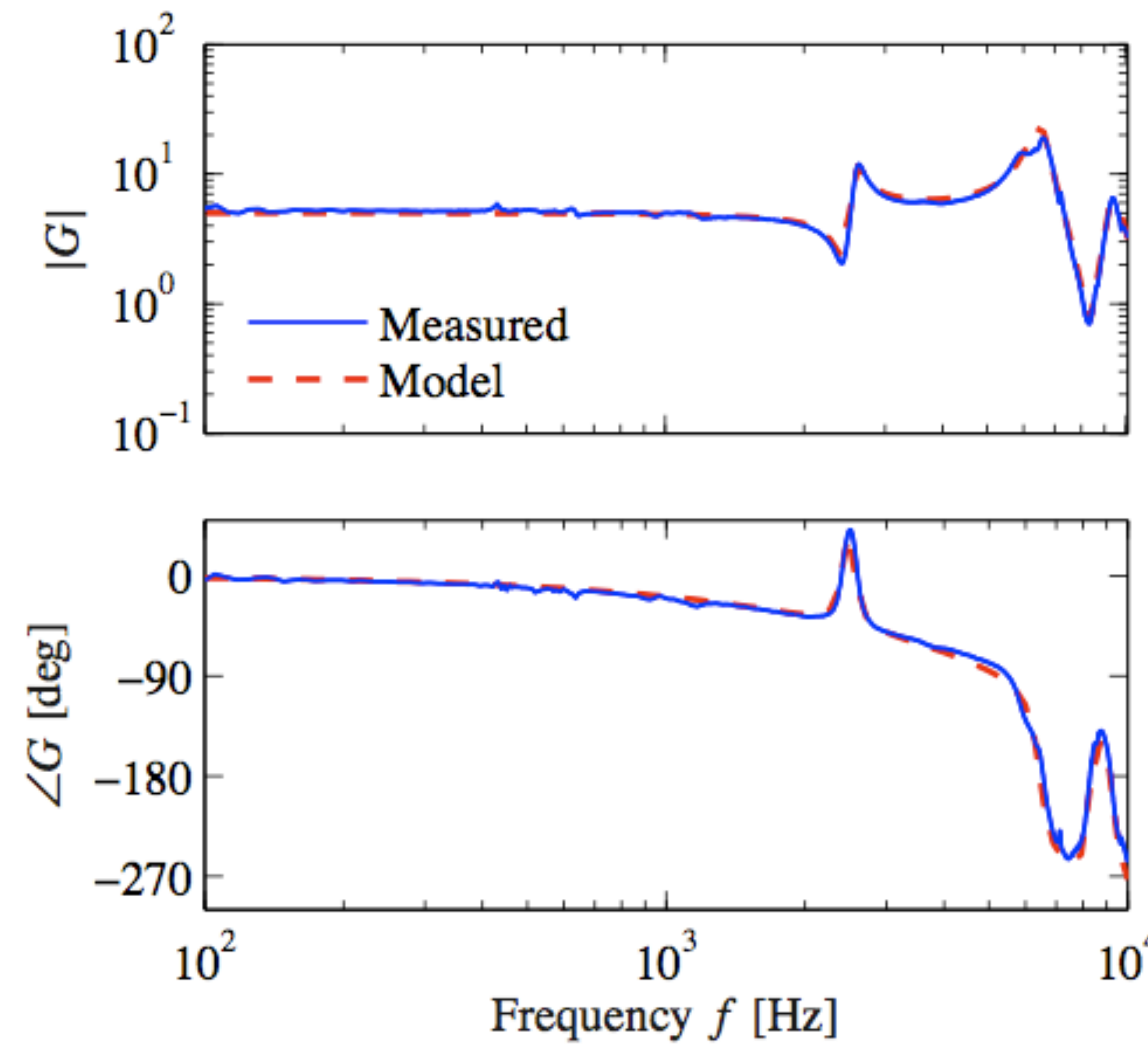
Example

$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}, \qquad a \ll b \ll \omega_0$$



Interpreting Bode Plots

- Build an input/output model by measuring the frequency response and fitting a transfer function to it.
- Spectrum analysis
- Zeros where the gain curve has local minima
- Poles where the gain curve has local maxima
- Relative damping ratios are adjusted to give a good fit. Time delay is adjusted to fit phase



$$G(s) = \frac{k\omega_2^2\omega_3^2\omega_5^2(s^2+2\zeta_1\omega_1 s+\omega_1^2)(s^2+2\zeta_4\omega_4 s+\omega_4^2)e^{-s\tau}}{\omega_1^2\omega_4^2(s^2+2\zeta_2\omega_2 s+\omega_2^2)(s^2+2\zeta_3\omega_3 s+\omega_3^2)(s^2+2\zeta_5\omega_5 s+\omega_5^2)},$$

Next Class

- Nyquist plots
- Stability margins