Lecture 15

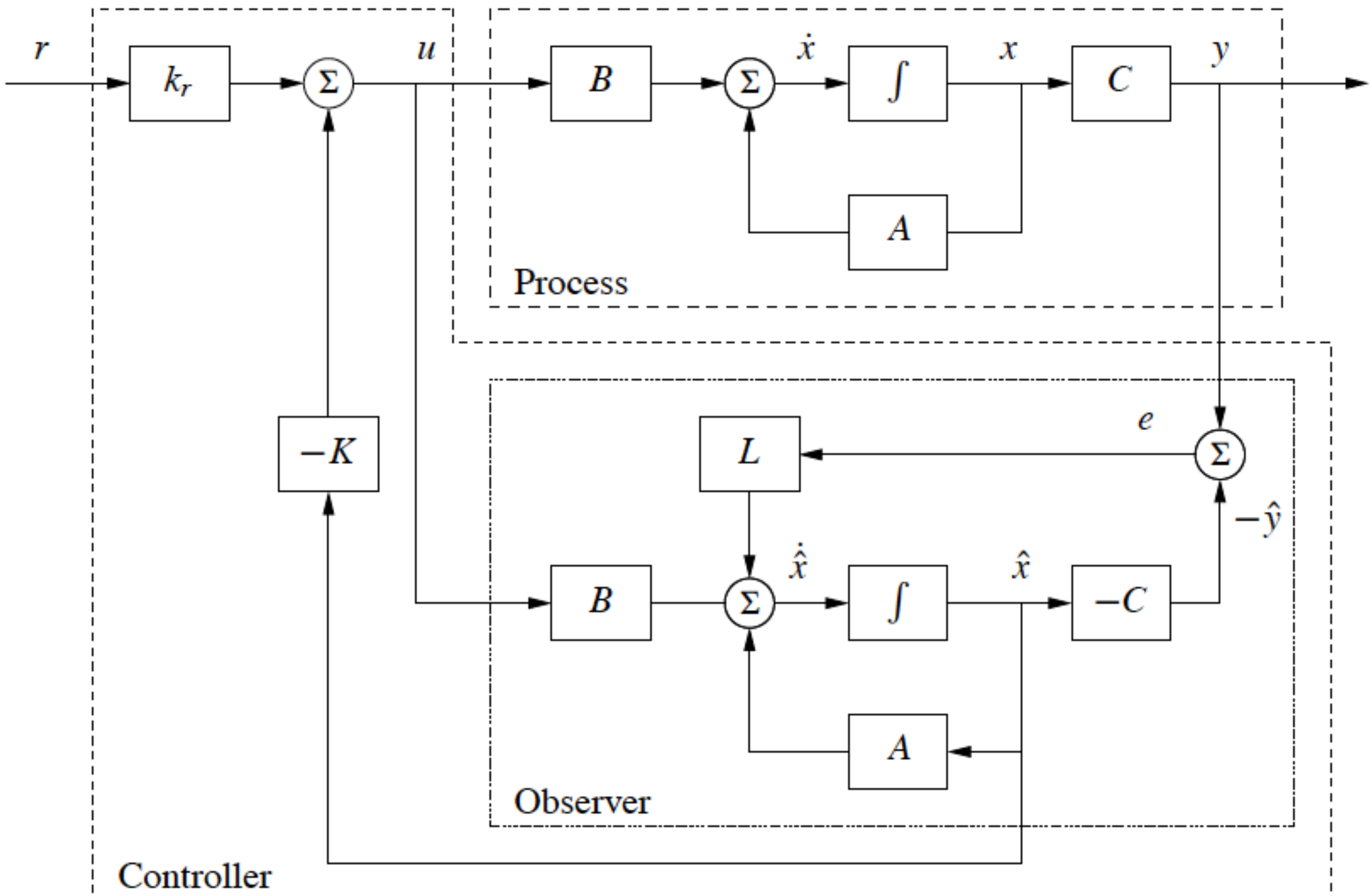
- why use transfer functions? -

Review

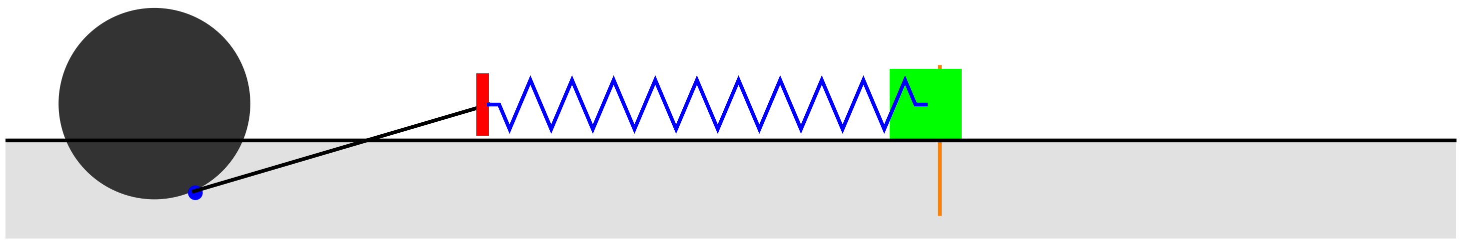
- Stability
  - Definitions
  - Analysis
- Reachability
  - Test for linear systems
- Observability
  - Test for linear systems
- Stabilization through state feedback

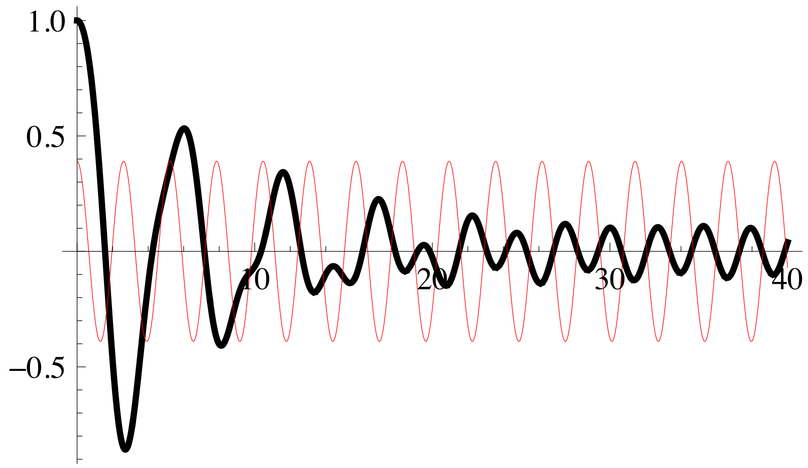
$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx. \qquad u = -K\hat{x} + k_r r$$

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$



Damped oscillator





$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u, \quad y = q$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q}/\omega_0 \end{pmatrix}$$

$$\frac{dx}{dt} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} x + \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

No input: $\lambda_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$

ζ : damping ratio

ω_0 : natural frequency

ω_d : damped frequency

$$\omega_d = \omega_0\sqrt{1 - \zeta^2}$$

Damped oscillator

overdamped

- If $\zeta > 1$

$$y(t) = \frac{\lambda_2 x_1(0) + x_2(0)}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} - \frac{\lambda_1 x_1(0) + x_2(0)}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$

where

$$\lambda_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1}, \quad \lambda_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1}$$

critically damped

- If $\zeta = 1$

$$y(t) = e^{-\omega_0 t}\left(x_1(0) + (x_2(0) + \omega_0 x_1(0))t\right)$$

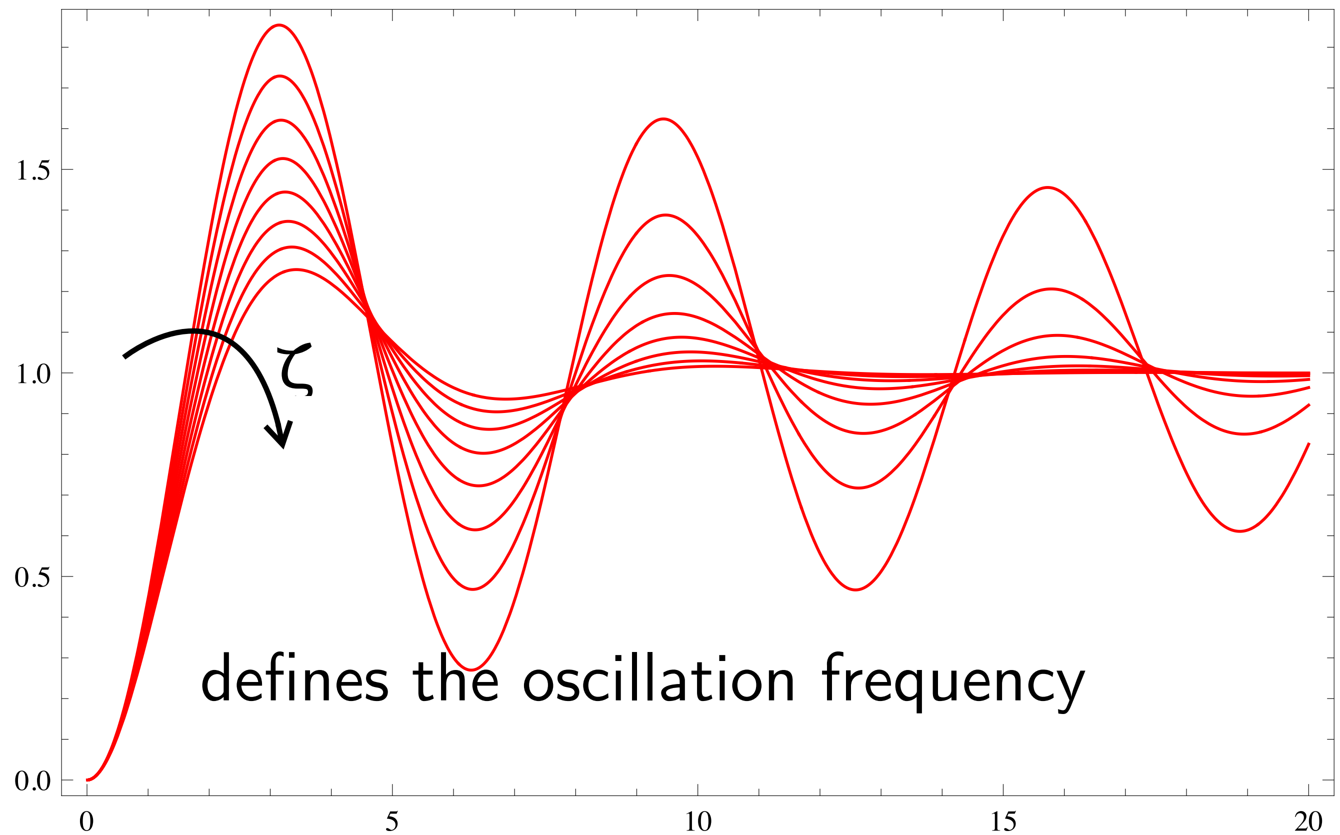
asymptotically stable as long as $\omega_0 > 0$

Damped oscillator

underdamped

- If $0 < \zeta < 1$

$$y(t) = e^{-\zeta\omega_0 t}\left(x_1(0)\cos\omega_d t + \left(\frac{\zeta\omega_0}{\omega_d}x_2(0) + \frac{1}{\omega_d}x_2(0)\right)\sin\omega_d t\right)$$

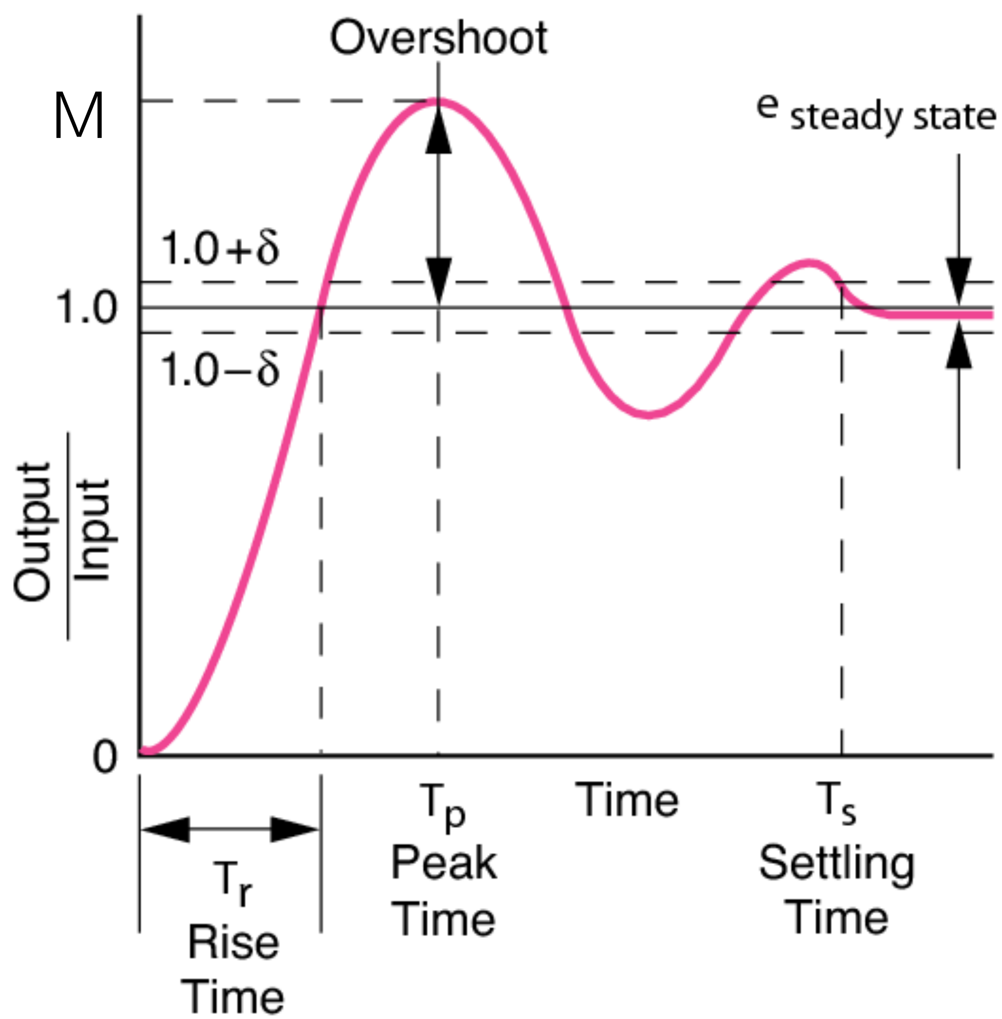


- If $\zeta \ll 1$ then $\omega_d \approx \omega_0$ defines the oscillation frequency

Performance (underdamped systems)

$$y(t) = e^{-\zeta\omega_0 t}\left(x_1(0)\cos\omega_d t + \left(\frac{\zeta\omega_0}{\omega_d}x_2(0) + \frac{1}{\omega_d}x_2(0)\right)\sin\omega_d t\right)$$

$$\lambda_{1,2} = \boxed{-\zeta\omega_0} \pm \omega_0\sqrt{\zeta^2 - 1}$$



Percentage of overshoot

$$M_p = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$P.O. = \frac{M_p - \text{final value}}{\text{final value}}100\%$$

$$= 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Peak time

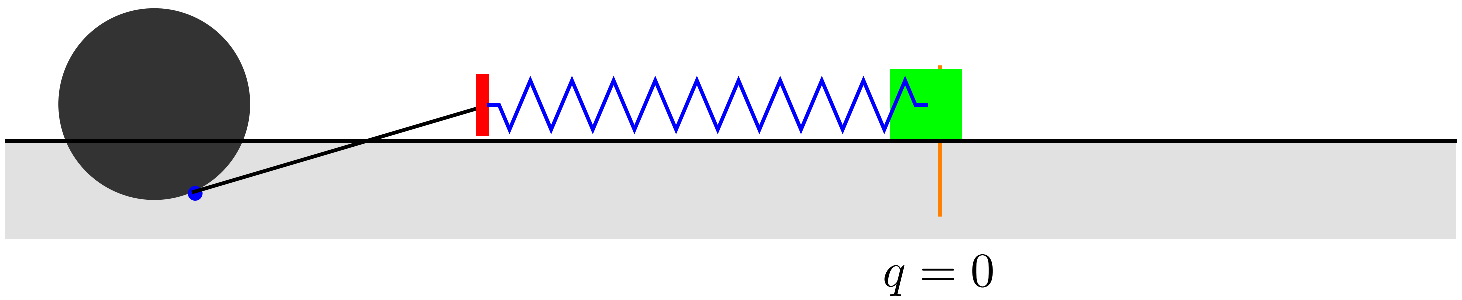
$$T_p = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}}$$

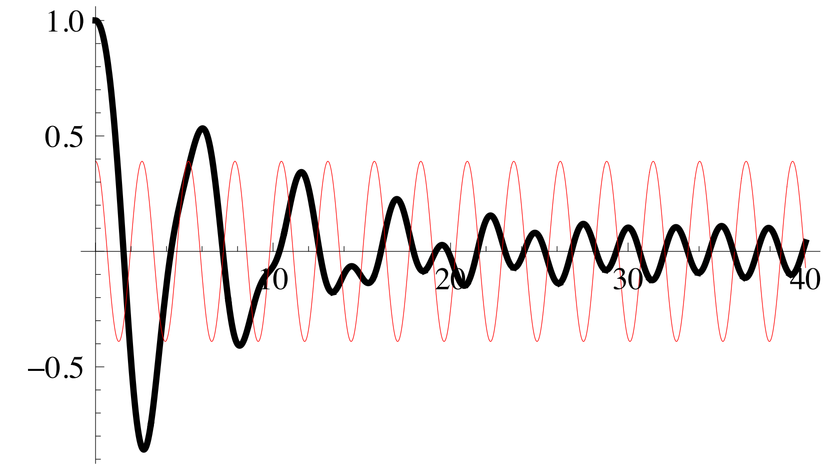
Settling time ($\boldsymbol{\delta} = 0.02$)

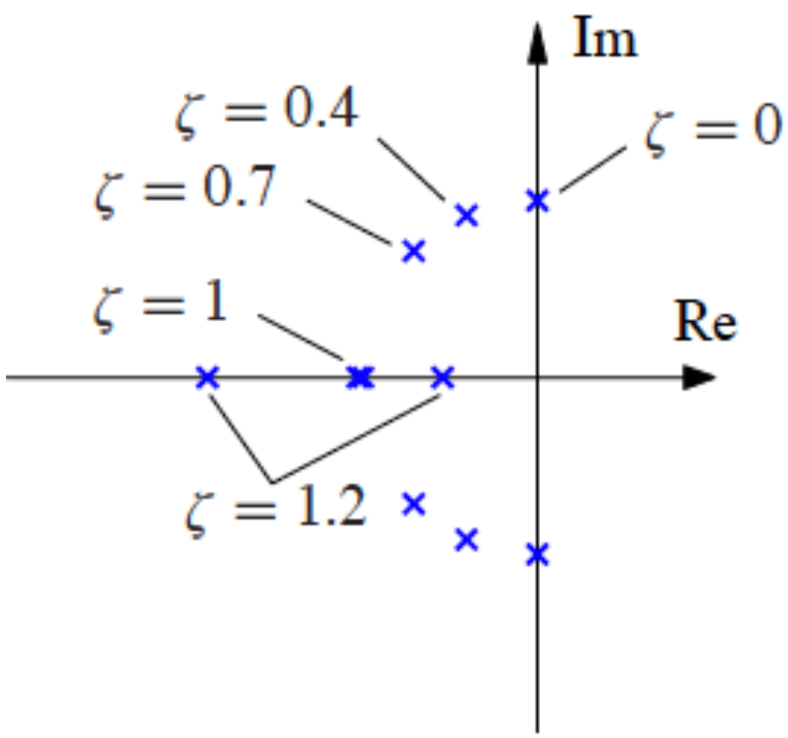
$$e^{\zeta\omega_0 T_s} < \delta$$

$$T_s \approx \frac{4}{\zeta\omega_0}$$

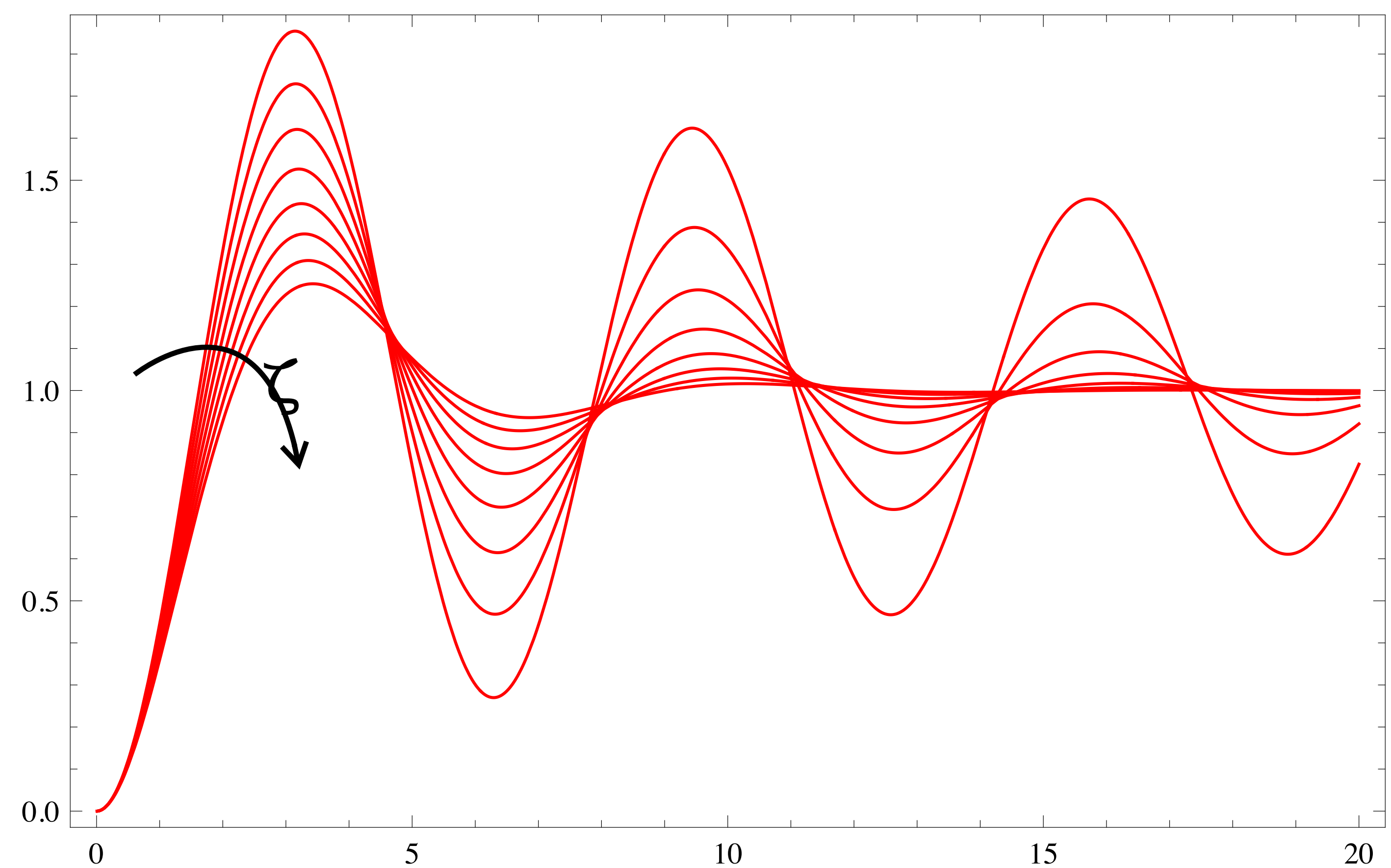
Damped oscillator







$$\lambda_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$



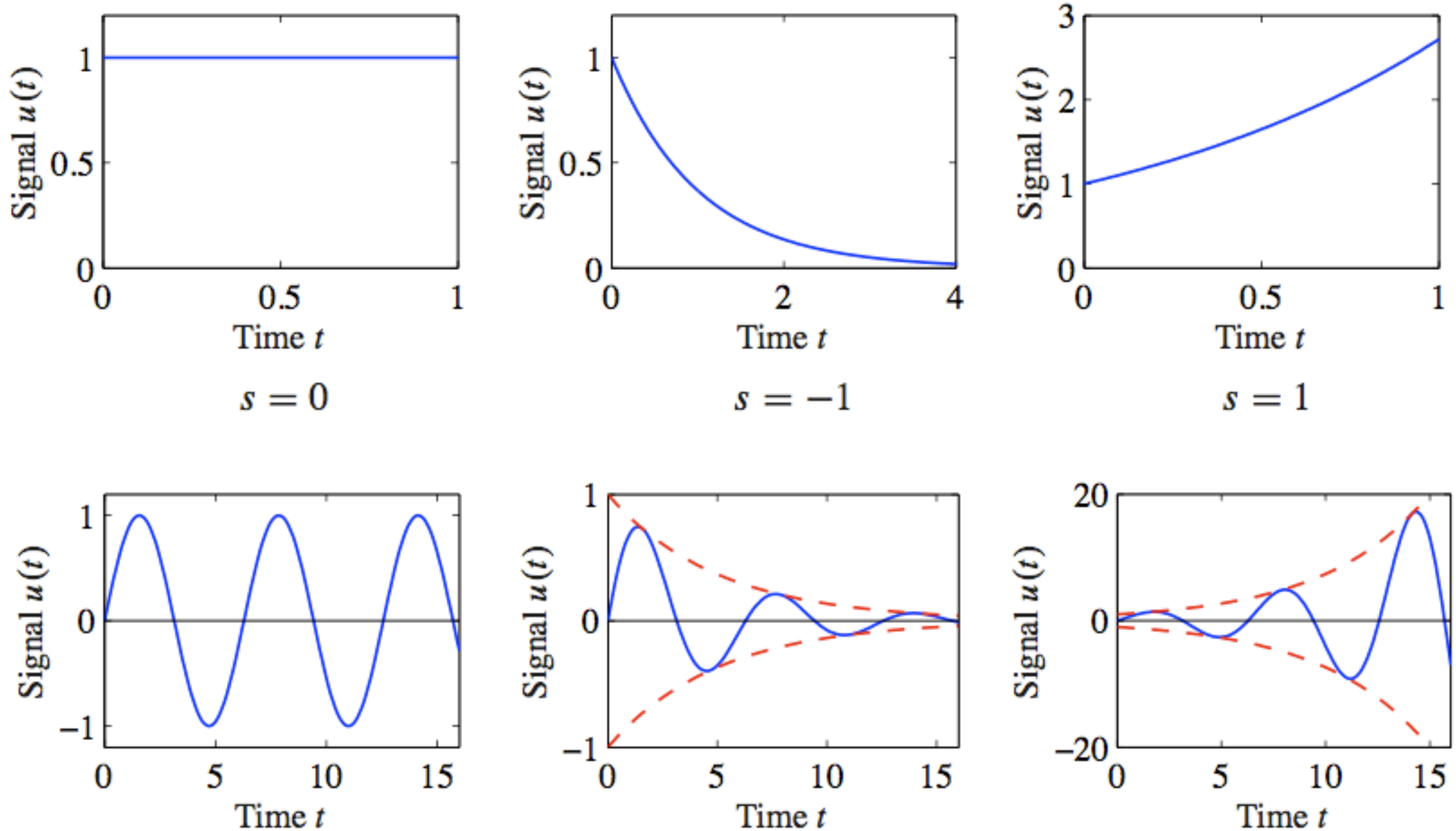
Today

- Input/output transfer function of a linear system
- Understand the relationships among
  - state-space model
  - frequency response (Bode plot)
  - transfer function
- Routh-Hurwitz stability criterion
- Canonical form realizations

Transmission of exponential signals

e^{st}, where $s = \sigma + i\omega$

$$e^{(\sigma+i\omega)t} = e^{\sigma t}e^{i\omega t} = e^{\sigma t}(\cos\omega t + i\sin\omega t),$$



$s = 0$ $\quad$ $s = -1$ $\quad$ $s = 1$

$s = i$ $\quad$ $s = -0.2 + i$ $\quad$ $s = 0.2 + i$

Derivation of the transfer function

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du.$$

$u(t) = e^{st}$ and assume that $s \neq \lambda_j(A)$, $j = 1, \dots, n$,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau}\, d\tau = e^{At}x(0) + e^{At} \int_0^t e^{(sI-A)\tau} B\, d\tau.$$

$$\begin{aligned} x(t) &= e^{At}x(0) + e^{At}(sI - A)^{-1}\left(e^{(sI-A)t} - I\right)B \\ &= e^{At}\left(x(0) - (sI - A)^{-1}B\right) + (sI - A)^{-1}Be^{st}. \end{aligned}$$

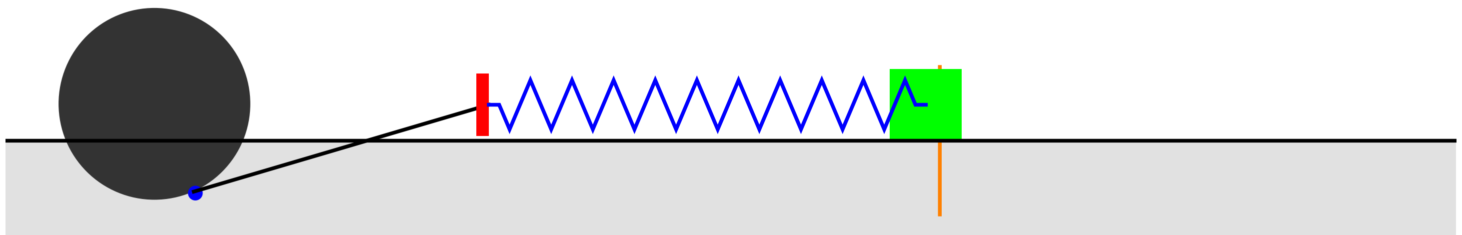
$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= \cancel{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)} + \left(C(sI - A)^{-1}B + D\right)e^{st}, \end{aligned}$$

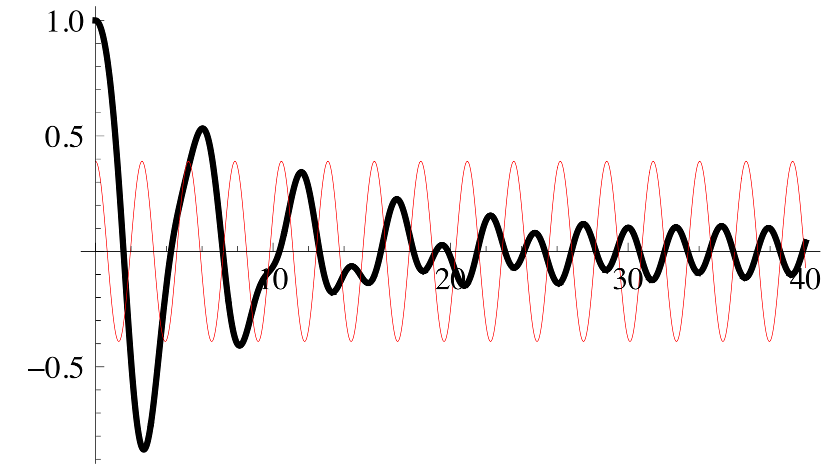
transient response
if the system is stable,
$y(t)$ does not depend on $x(0)$

pure exponential response

$$\boxed{G_{yu}(s) = C(sI - A)^{-1}B + D}$$

Damped oscillator





$$\frac{dx}{dt} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} x + \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

$$\begin{aligned} G_{yu}(s) &= C(sI - A)^{-1}B \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -\omega_0 \\ \omega_0 & s + 2\zeta\omega_0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \begin{pmatrix} s + 2\zeta\omega_0 & \omega_0 \\ -\omega_0 & s \end{pmatrix} \right) \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} \\ &= \frac{k\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \end{aligned}$$

Second-order linear system

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

$$G_{yu}(s) = C(sI - A)^{-1}B$$
$$= \frac{k\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$s = \sigma + \omega i$$

steady-state response to a step function, we set $s=0$

$$u = 1 \qquad \Longrightarrow \qquad y = G_{yu}(0)u = k.$$

steady-state response to a sinusoid

$$u = \sin \omega t = \frac{1}{2}\left(ie^{-i\omega t} - ie^{i\omega t}\right),$$
$$y = \frac{1}{2}\left(iG_{yu}(-i\omega)e^{-i\omega t} - iG_{yu}(i\omega)e^{i\omega t}\right).$$

$$G(i\omega) = \frac{k\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = Me^{i\theta},$$

$$y = \frac{1}{2}\left(i(Me^{-i\theta})e^{-i\omega t} - i(Me^{i\theta})e^{i\omega t}\right)$$
$$= M \cdot \frac{1}{2}\left(ie^{-i(\omega t+\theta)} - ie^{i(\omega t+\theta)}\right)$$
$$= M\sin(\omega t + \theta)$$

Type	ODE	Transfer Function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s
First-order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
PID controller	$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \frac{k_i}{s}$
Time delay	$y(t) = u(t - \tau)$	$e^{-\tau s}$

Gains, poles and zeros

$$G(s) = \frac{b(s)}{a(s)},$$

roots of $b(s)$ are called the zeros

roots of $a(s)$ are called the poles

$$G_{yu}(s) = C(sI - A)^{-1}B + D,$$

Should the pair of zeros lie directly upon the imaginary axis the system does not respond at all to sinusoidal excitation at that frequency.

$\Rightarrow$ response is zero at the frequency of the zero

Poles of the transfer function are the eigenvalues of the matrix A

$\Rightarrow$ tf stable if all of its poles have negative real part

Routh-Hurwitz Criterion

$$G(s) = \frac{b(s)}{a(s)}, \qquad a(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & \dfrac{a_{n-1}a_{n-2} - a_{n-3}a_n}{a_{n-1}} & \dfrac{a_{n-1}a_{n-4} - a_{n-5}a_n}{a_{n-1}} & \cdots & \\
\vdots & \vdots & & & \\
s^1 & & & & \\
s^0 & & & &
\end{array}$$

Transfer function

- A compact description of the input/output
- Use tf to connect systems (block algebra)
- Response of the system to an exponential input

Routh-Hurwitz Criterion

$$G(s) = \frac{b(s)}{a(s)}, \qquad a(s) = s^4 + 8s^3 + 18s^2 + 16s + 5$$

$$\begin{array}{c|ccc} s^4 & \boxed{1} & 18 & 5 \\ s^3 & 8 & 16 & \\ s^2 & 16 & 5 & \\ s^1 & 27/2 & & \\ s^0 & 5 & & \end{array}$$

Number of sign changes equals number of roots in RHP

Routh-Hurwitz Criterion

$$G(s) = \frac{b(s)}{a(s)}, \qquad a(s) = 3s^4 + 10s^3 + 5s^2 + 5s + 2$$

$$\begin{array}{c|ccc}
s^4 & \boxed{3} & 5 & 2 \\
s^3 & \boxed{10} & 5 & \\
s^2 & \boxed{7/2} & 2 & \\
s^1 & \boxed{-1/7} & & \\
s^0 & \boxed{2} & &
\end{array}
\qquad
\begin{array}{c|ccc}
s^4 & \boxed{3} & 5 & 2 \\
s^3 & \boxed{2} & 1 & \\
s^2 & \boxed{7/2} & 2 & \\
s^1 & \boxed{-1/7} & & \\
s^0 & \boxed{2} & &
\end{array}$$

2 poles in RHP

Routh-Hurwitz Criterion

$$G(s) = \frac{b(s)}{a(s)}, \qquad a(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$$

$$\begin{array}{c|ccc}
s^5 & 1 & 4 & 3 \\
s^4 & 1 & 24 & 63 \\
s^3 & -20 & -60 & \\
s^2 & -21 & -63 & \\
s^1 & \boxed{0} & & \\
s^0 & & &
\end{array}$$

Nothing can be said!

Realization of a transfer function H(s)

- A state space model $(A,\ B,\ C,\ D)$ if its transfer function coincides with $H(s)$:

$$C(sI - A)^{-1}B + D = H(s)$$

- A transfer function $H(s)$ is realizable if there exists a state space model $(A,\ B,\ C,\ D)$ with transfer function $H(s)$

$H(s)$ is realizable $\Rightarrow$ $H(s)$ is rational and proper

$$\boxed{\begin{array}{lllll} \hat{G}(s) & = & \frac{s}{s^2+1} & \text{rational,} & \text{strictly proper} \\ \hat{G}(s) & = & \frac{s^2}{s^2+1} & \text{rational,} & \text{proper} \end{array}}$$

$$\begin{array}{lllll} \hat{G}(s) & = & \frac{s^3}{s^2+1} & \text{rational,} & \text{not proper} \\ \hat{G}(s) & = & \frac{e^{-s}}{s^2+1} & \text{not rational} & \end{array}$$

Controllable canonical form realization

- Suppose that $H(s)$ can be expressed as

$$H(s) = \frac{\beta_{n-1}s^{n-1} + \ldots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0} + d$$

- Example:

$$\begin{aligned} H(s) &= \frac{4s+8}{-2s+10} \\ &= \frac{-2(-2s+10)+8+20}{-2s+10} \\ &= \frac{28}{-2s+10} - 2 \\ &\boxed{= \frac{-14}{s-5} - 2} \end{aligned}$$

rational, strictly proper

Controllable canonical form realization

The transfer function $H(s)$ can be expressed as

$$
\begin{aligned}
H(s) &= \frac{\beta_{n-1}s^{n-1} + \ldots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0} + d \\
&= C(s_I - A)^{-1}B + D
\end{aligned}
$$

$$
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & & \ddots & & \vdots \\
\vdots & \vdots & & & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
-\alpha_0 & -\alpha_1 & -\alpha_2 & \ldots & -\alpha_{n-2} & -\alpha_{n-1}
\end{pmatrix}
\qquad
B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}
$$

$$
C = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \ldots & \beta_{n-2} & \beta_{n-1} \end{pmatrix} \qquad D = d
$$

Example

$$G(s) = \frac{s+2}{s^2+2s+1}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} x(t)$$

verify: $C(s_I - A)^{-1}B = \dfrac{s+2}{s^2+2s+1}$

note: $M = [B \ AB] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ Always controllable!

Example

$$H(s) = \frac{s^2 + 3s + 3}{s^2 + 2s + 1}$$

$$= \frac{s + 2}{s^2 + 2s + 1} + 1$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} x(t) + \boxed{[1]u(t)}$$

Observable canonical form realization

The transfer function $H(s)$ can be expressed as

$$
\begin{aligned}
H(s) &= \frac{\beta_{n-1}s^{n-1} + \ldots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0} + d \\
&= C(s_I - A)^{-1}B + D
\end{aligned}
$$

$$
A = \begin{pmatrix} 0 & 0 & \ldots & 0 & -\alpha_0 \\ 1 & 0 & \ldots & 0 & -\alpha_1 \\ 0 & 1 & \ddots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & -\alpha_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \beta_{n-1} \end{pmatrix}
$$

$$
C = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \qquad D = d
$$

Next

- Root locus
- Bode plots