Lecture 11

- can I get there? -

Analysis and design of systems

non-linear systems

Lyapunov stability (general theory)

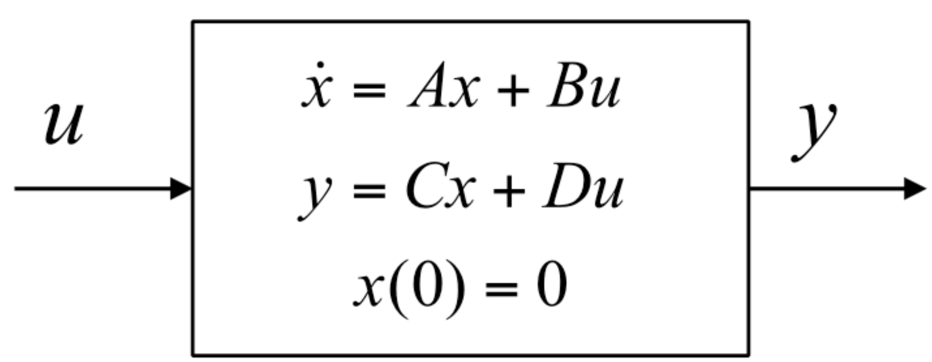
Theorems (Lyapunov/Lasalle)

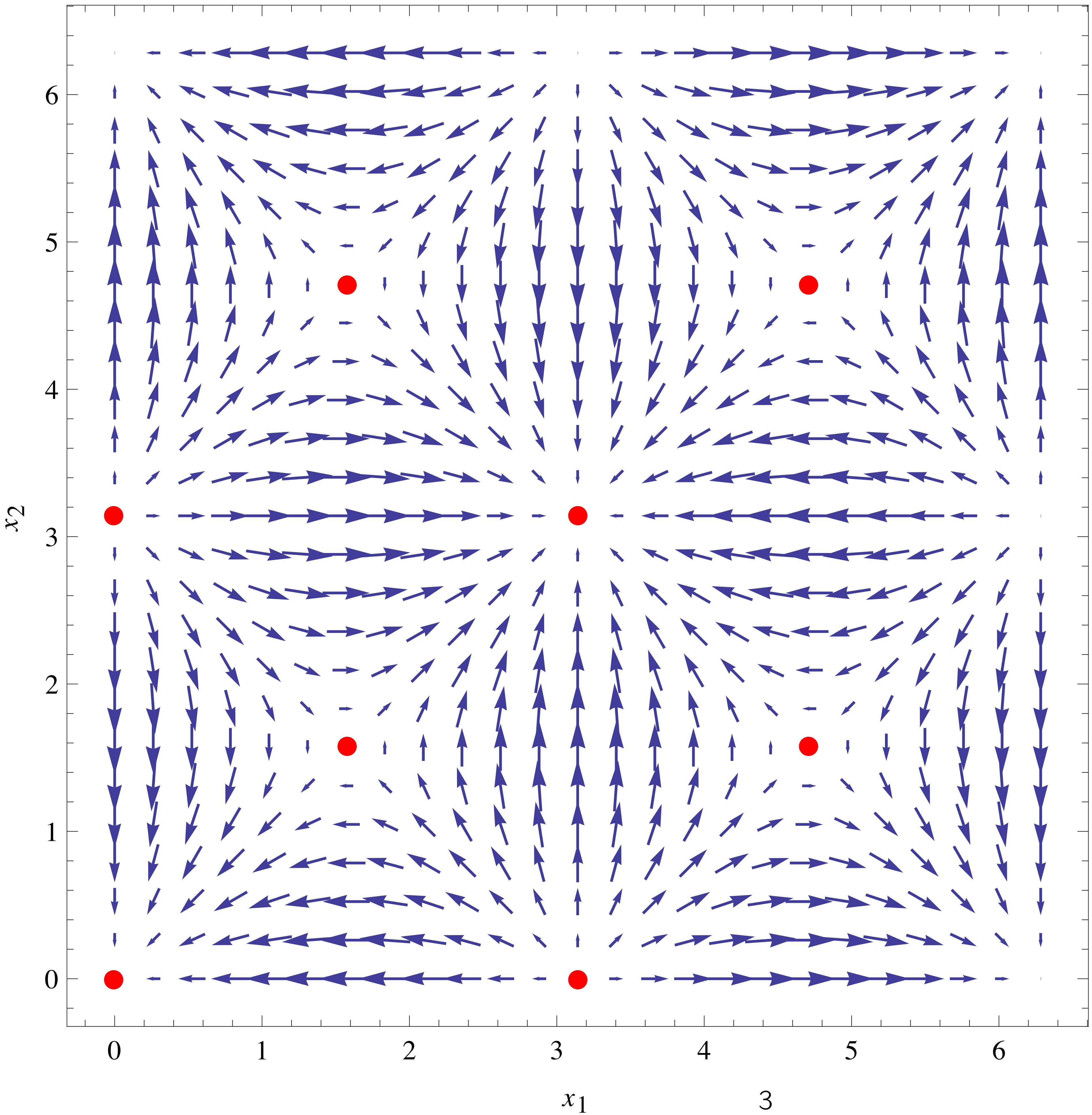
↓

linear systems

Lyapunov stability (special case)

Theorems (eigenvalues)





$$\dot{x}_1 = -\sin x_1 \cos x_2$$

$$\dot{x}_2 = -\cos x_1 \sin x_2$$

Analysis and design of systems

non-linear systems

Lyapunov stability (general theory)

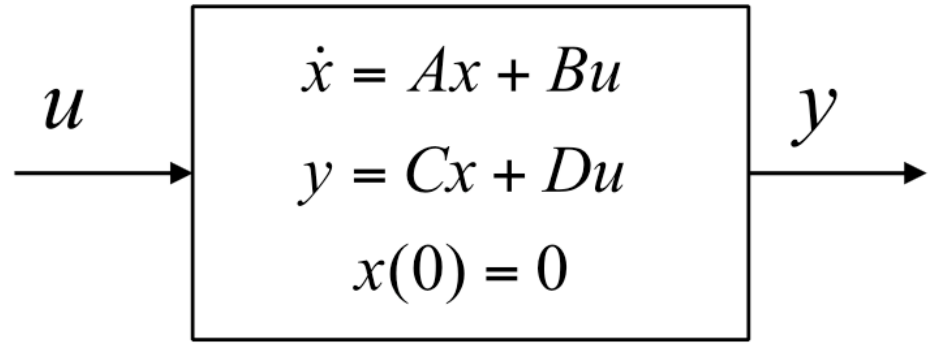
Theorems (Lyapunov/Lasalle)

linear systems

Lyapunov stability (special case)

Theorems (eigenvalues)

local description of non-linear dynamics



Properties of linear systems

- Linearity with respect to initial conditions and inputs
- Stability characterized by eigenvalues
- Many applications and design tools available

Today

- Define reachability of a system
- Give test for reachability of linear systems
- State feedback for linear systems

Design concepts

- Stabilization: stabilize the system around an equilibrium point

  Given an equilibrium point, find a "control law" $u = \alpha(x)$

  such that $\lim_{t \to \infty} x(t) = x_e$ for all $x(0) \in \mathbb{R}^n$.

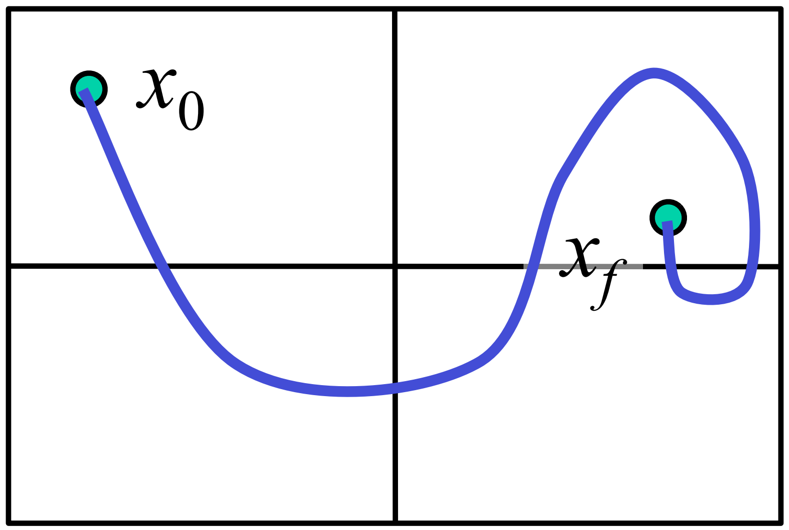
  Control law will depend on the state!

- Reachability: steer the system between two points

  Given $x_0 = x(0)$ and $x_f = x(T)$, find an input such that

  $$\dot{x} = f(x, u(t))$$

  takes x_0 to x_f.



- caveat -

you need to know if a system is reachable before you try to stabilize it

Reachability test

Definition
A linear system is reachable if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and $u : [0, T] \mapsto \mathbb{R}$ such that the corresponding solution satisfies

$$x(0) = x_0 \text{ and } x(T) = x_f\,.$$

Reachability test (for linear system)
Consider a linear system of the form

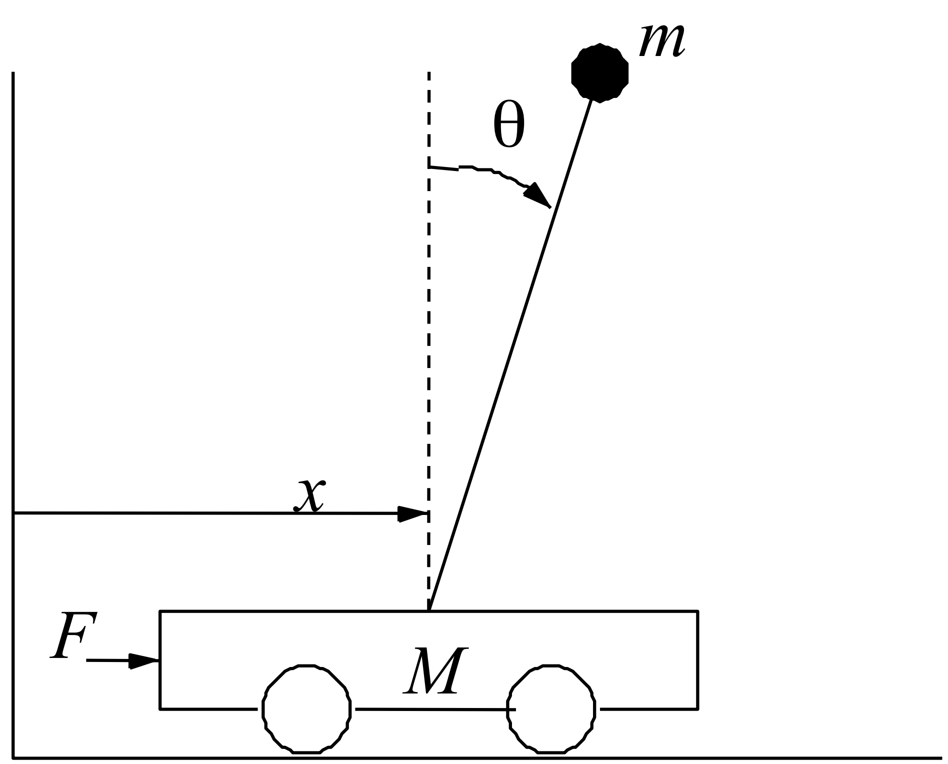
$$\frac{dx}{dt} = Ax + Bu$$

The system is reachable if and only if the reachability matrix

$$W_r = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is invertible. That is, if and only if W_r is of full rank.

Example: inverted pendulum



Question: can we locally control the position of the cart by proper choice of input?

Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2gl^2}{J(M+m)+Mml^2} & \dfrac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \dfrac{mgl(M+m)}{J(M+m)+Mml^2} & \dfrac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J+ml^2}{J(M+m)+Mml^2} \\ \dfrac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Example: simplified dynamics

$$
\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2gl^2}{q} & \cancelto{0}{\dfrac{-(J+ml^2)b}{q}} & 0 \\ 0 & \dfrac{mgl(M+m)}{q} & \cancelto{0}{\dfrac{-mlb}{q}} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J+ml^2}{q} \\ \dfrac{ml}{q} \end{bmatrix} u
$$

$$q = J(M+m) + Mml^2$$

Simplify by setting b = 0, m=l=g=l

$$
\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2gl^2}{q} & \dfrac{-(J+ml^2)b}{q} & 0 \\ 0 & \dfrac{mgl(M+m)}{q} & \dfrac{-mlb}{q} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J+ml^2}{q} \\ \dfrac{ml}{q} \end{bmatrix} u
$$

$$q = J(M+m) + Mml^2$$

Example: Reachability matrix

$$\begin{bmatrix} A^3B & A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 1/q & 0 & (J+1)/q & 0 \\ (M+1)/q & 0 & 1/q & 0 \\ 0 & 1/q & 0 & (J+1)/q \\ 0 & (M+1)/q & 0 & 1/q \end{bmatrix}$$

Full rank!