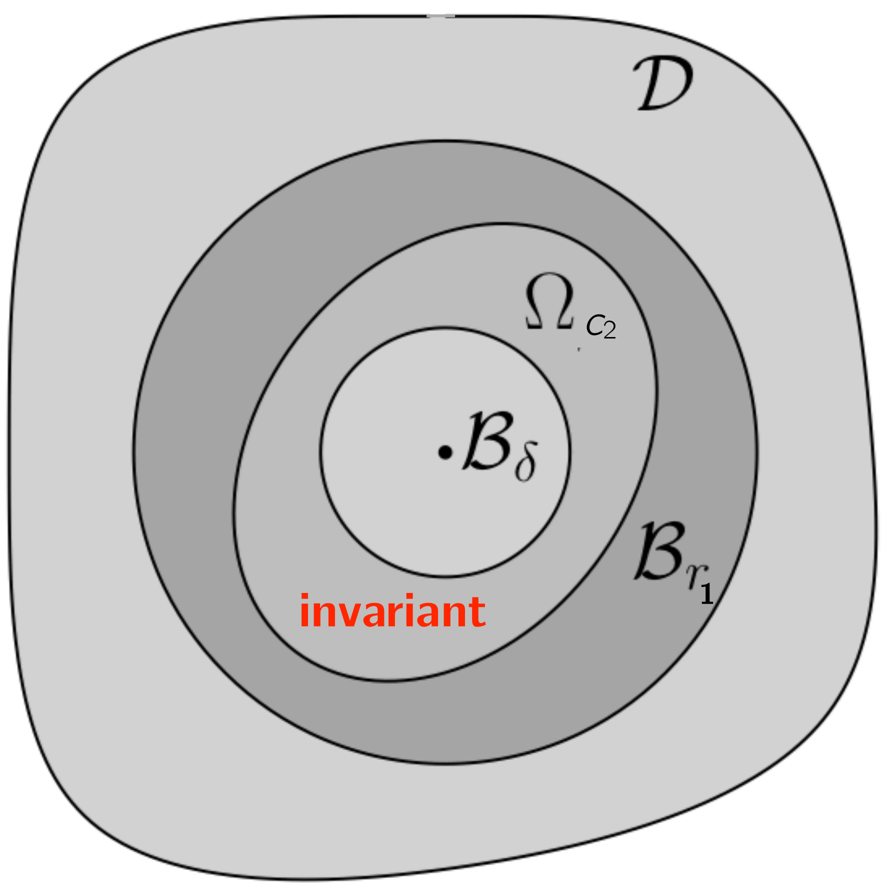
Lecture 6
- how to apply stability result? -

Lyapunov's theorem on stability

Let $x_e = 0$ be an equilibrium point. Assume that there exists an open set D with $x_e \in D$ and a continuously differentiable function $V : D \to R$ such that:

1. $V(0) = 0$,
2. $V(x) > 0 \forall x \in D \setminus \{0\}$, and
3. $\dfrac{\partial V}{\partial x}(x) f(x) \leq 0 \ \forall x \in D$

Then $x_e = 0$ is a stable equilibrium.



Proof

Choose $r_1 \in (0, \varepsilon)$ such that $B_{r_1} \subseteq D$

Let $c_1 = \min_{x \in S_{r_1}} V(x)$

Choose $c_2 \in (0, c_1)$

Then $\Omega_{c_2} \subseteq B_{r_1} \subseteq B_\varepsilon$.

Choose $\delta > 0$ such that $B_\delta \in \Omega_{c_2}$

$$B_r = \{x \in R^n : |x| < r\}$$
$$S_r = \{x \in R^n : |x| = r\}$$
$$\Omega_r = \{x \in R^n : V(x) < r\}$$

Example

Consider a mass-spring-damper system

$$m(1+q^2)\ddot{q} + b\dot{q} + k_0 q + k_1 q^3 = u$$

Let $x_1 = q$ and $x_2 = \dot{q}$

Then

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{u - k_0 x_1 - k_1 x_1^3 - b x_2}{m(1+x_1^2)}\end{aligned}$$

Given the desired state $q_d = x_d$, define the error as $e = q_d - q = x_d - x_1$

Consider the Lyapunov function candidate

$$\begin{aligned}V(x) &= \frac{1}{2}(\dot{e} + \alpha e)^2 \\ &= \frac{1}{2}\left((\dot{x}_d - \dot{x}_1) + \alpha(x_d - x_1)\right)^2 \\ &= \frac{1}{2}\left((\dot{x}_d - x_2) + \alpha(x_d - x_1)\right)^2\end{aligned}$$

Exponential stability

To prove exponential stability we need to guarantee that

$$\dot{V}(x) = -kV(x) \text{ for some } k > 0$$

Computing the derivative of $V(x)$ we get

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial e}\frac{de}{dt} \\ &= (\dot{e} + \alpha e)(\ddot{e} + \alpha \dot{e})\end{aligned}$$

Since $V(x) = \frac{1}{2}(\dot{e} + \alpha e)^2$, we know that $\dot{V}(x) = -kV(x)$ if

$$(\ddot{e} + \alpha \dot{e}) = -\frac{k}{2}(\dot{e} + \alpha e)$$

Substituting $e = (x_d - x_1), \dot{e} = (\dot{x}_d - x_2),$ and $\ddot{e} = \ddot{x}_d - \dot{x}_2$

$$\ddot{x}_d - \frac{u - k_0 x_1 - k_1 x_1^3 - b x_2}{m(1 + x_1^2)} + \alpha(\dot{x}_d - x_2) = -\frac{k}{2}(\dot{x}_d - x_2 + \alpha(x_d - x_1))$$

Exponential stability

Solving for u yields the control law

$$\begin{aligned} u &= m(1+x_1^2)\left(\frac{k}{2}(\dot{x}_d - x_2 + \alpha(x_d - x_1)) + \alpha(\dot{x}_d - x_2) + \ddot{x}_d\right) + k_0 x_1 + k_1 x_1^3 + b x_2 \\ &= m(1+x_1^2)\left(\frac{\alpha k}{2}(x_d - x_1) + (\alpha + \frac{k}{2})(\dot{x}_d - x_2) + \ddot{x}_d\right) + k_0 x_1 + k_1 x_1^3 + b x_2 \end{aligned}$$

where $k > 0$ and $\alpha > 0$ are tunable control parameters

Remarks:

- The input u guarantees global exponential stability
- Upon substitution into the time derivative yields $\dot{V} = -kV(x)$
- This first order differential equation which has solution

$$V = V(0)e^{-kt} = (\dot{e}(0) + \alpha e(0))e^{-kt} = (\dot{x}_d - x_2 + \alpha(x_d - x_1))e^{-kt}$$

- The error and error rate exponentially decay to zero