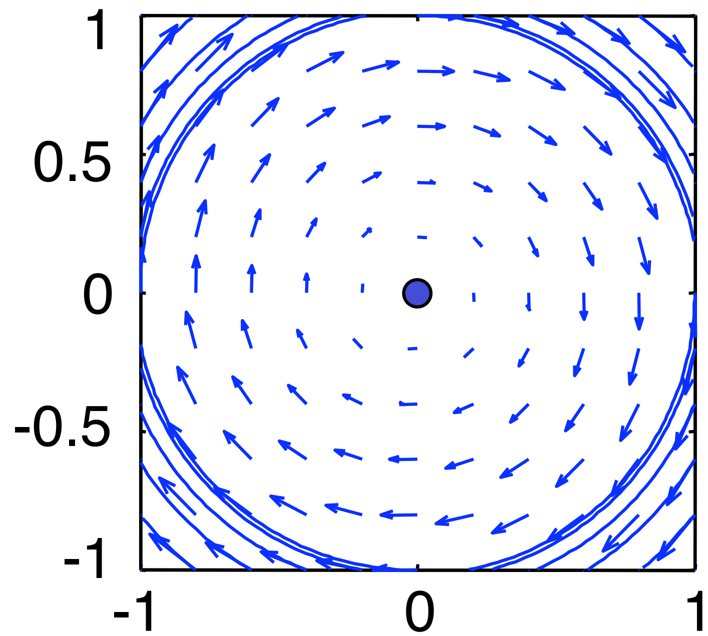
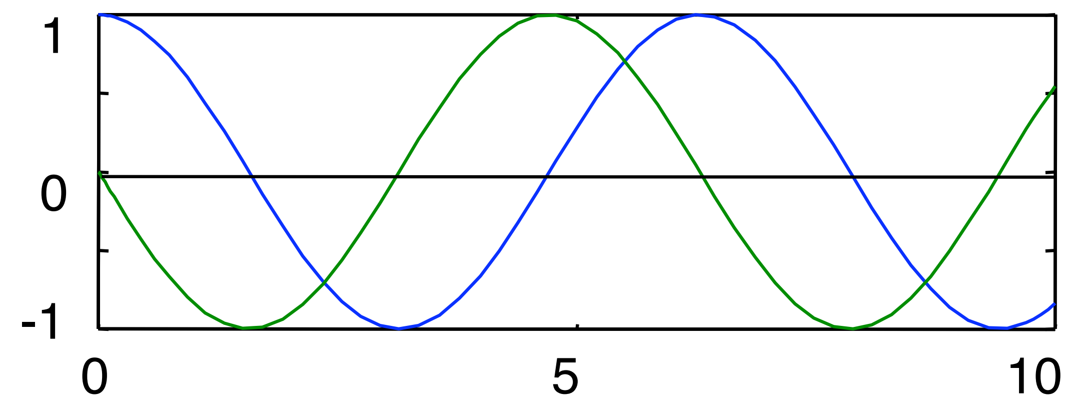
Lecture 4
- is an equilibrium stable? -

Stability of equilibrium points

Stable (in the sense of Lyapunov)

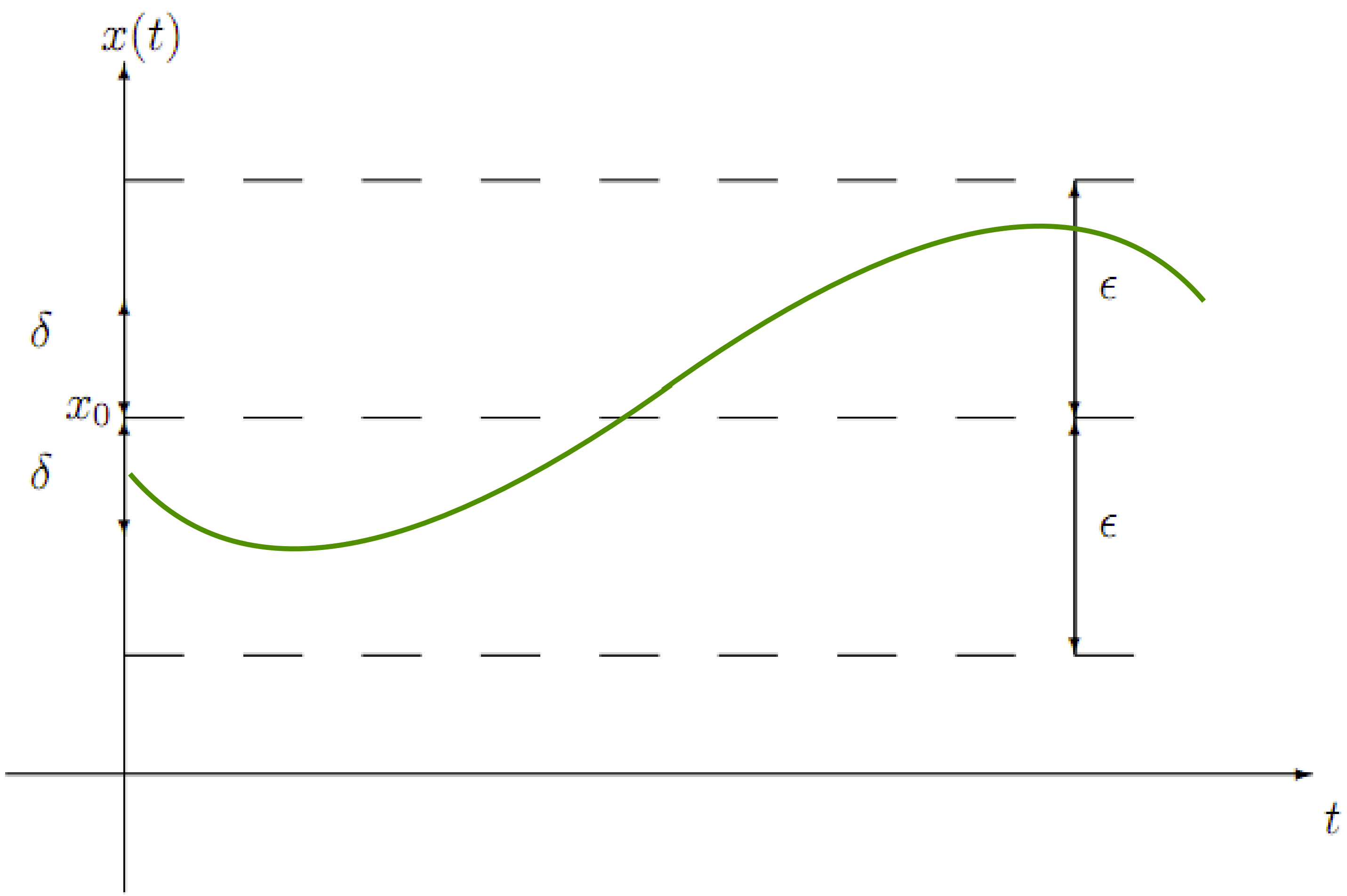




e.p. (equilibrium point) is a center

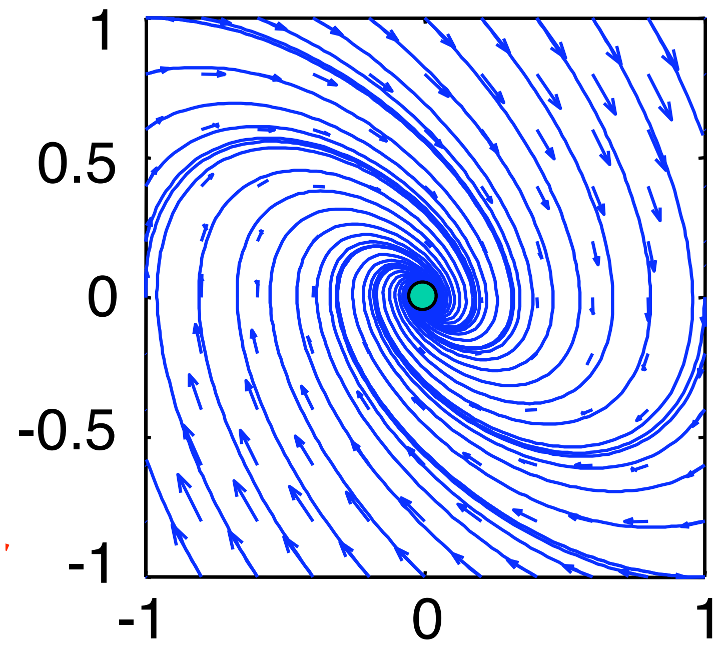
For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that:

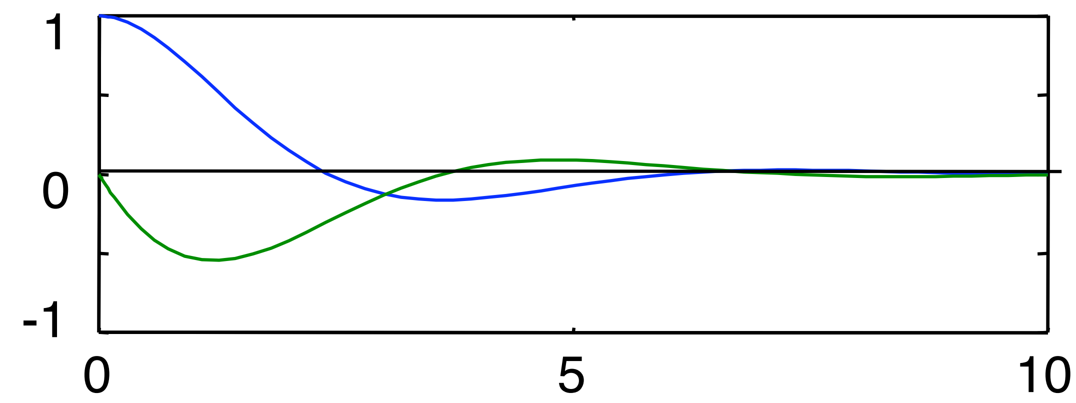
If $\|x(0) - x_e\| < \delta$, then $\|x(t) - x_e\| < \varepsilon$, for every $t \geq 0$.



If the condition is not satisfied, then the e.q. is unstable

Asymptotically Stable





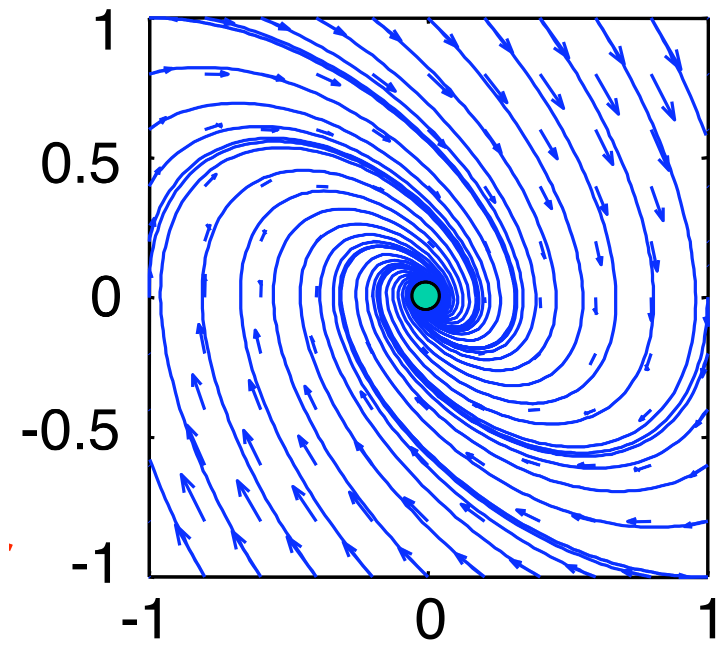
e.p. is an attractor or sink

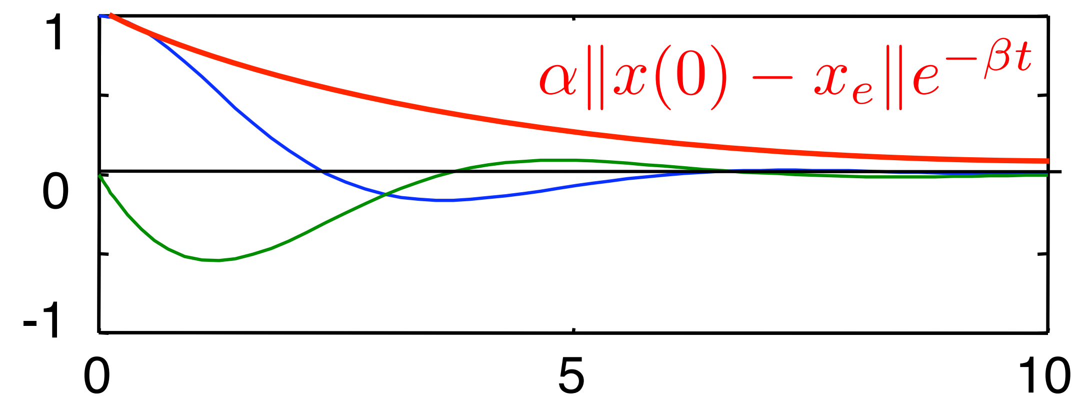
1. Lyapunov stable

2. There exists a $\delta > 0$ such that:

$$\text{If } \|x(0) - x_e\| < \delta, \text{ then } \lim_{t\to\infty} \|x(t) - x_e\| = 0$$

Exponentially stable





Any e.p. that is exponentially stable is also asymptotically stable.

1. Asymptotically stable

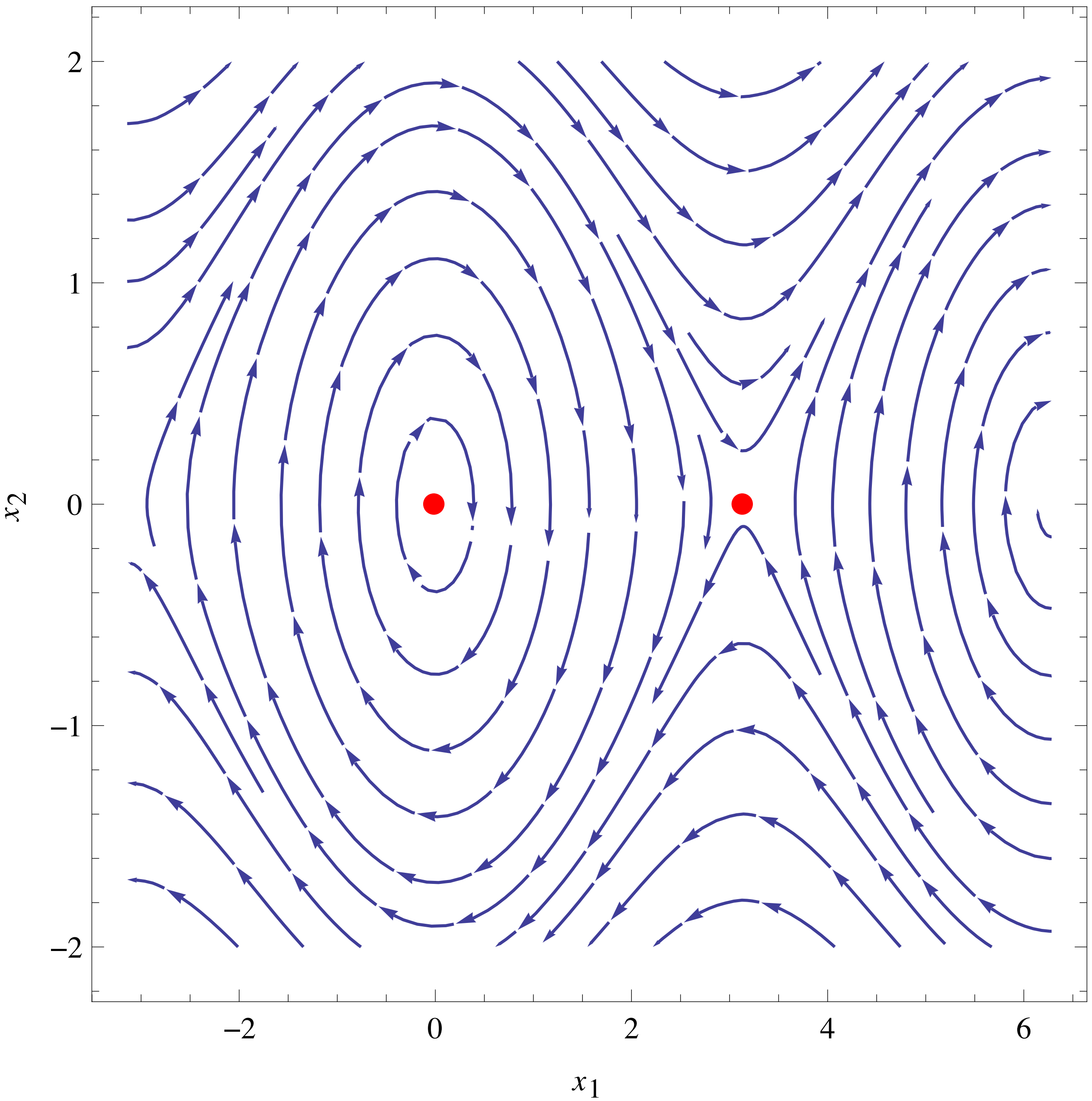
2. There exists $\alpha, \beta, \delta > 0$ such that:

$$\text{If } \|x(0) - x_e\| < \delta, \text{ then} \|x(t) - x_e\| \leq \alpha \|x(t) - x_e\| e^{-\beta t}, \text{ for } t \geq 0$$

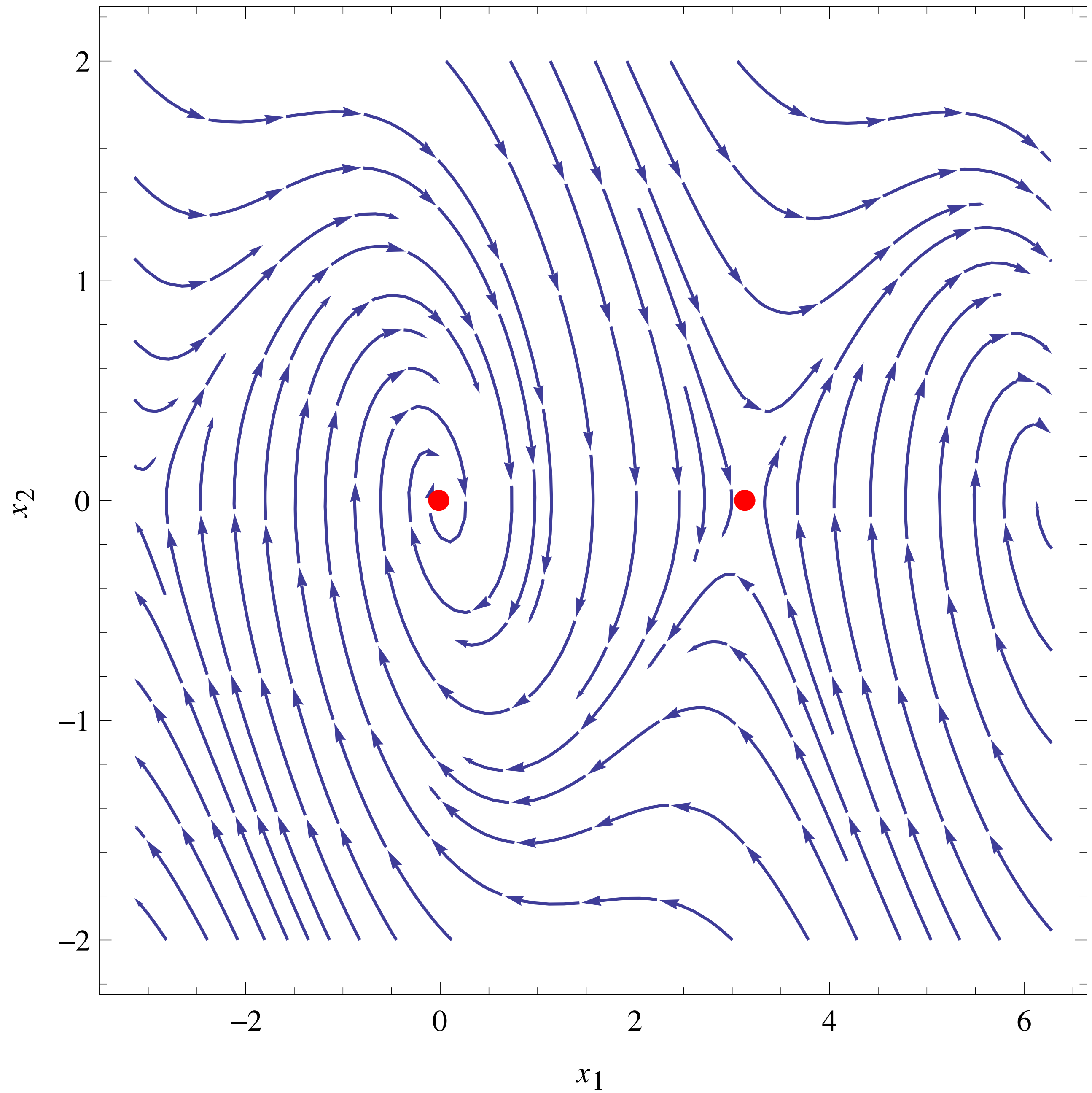
Pendulum with friction

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

$k = 0$

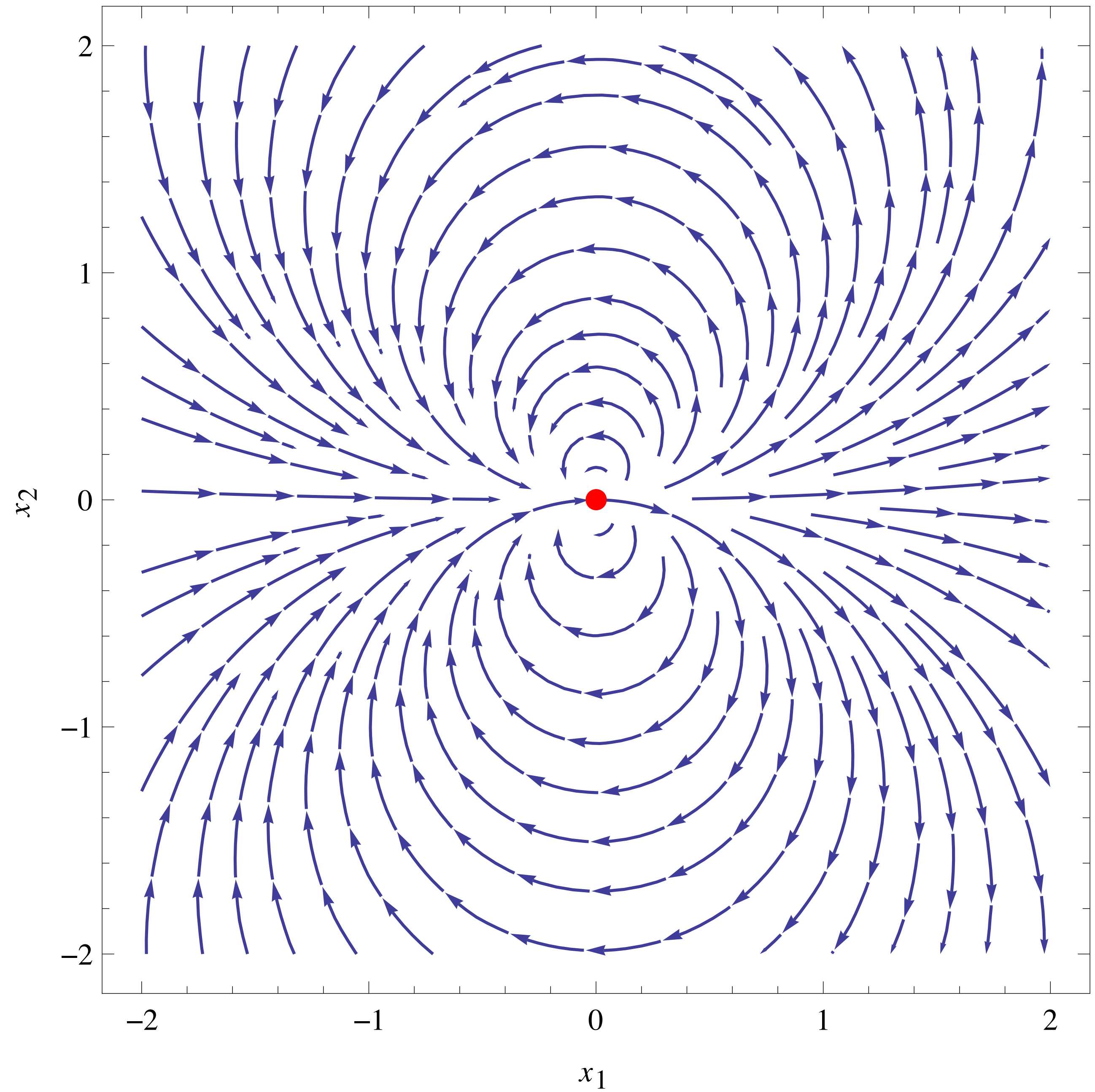


$k > 0$



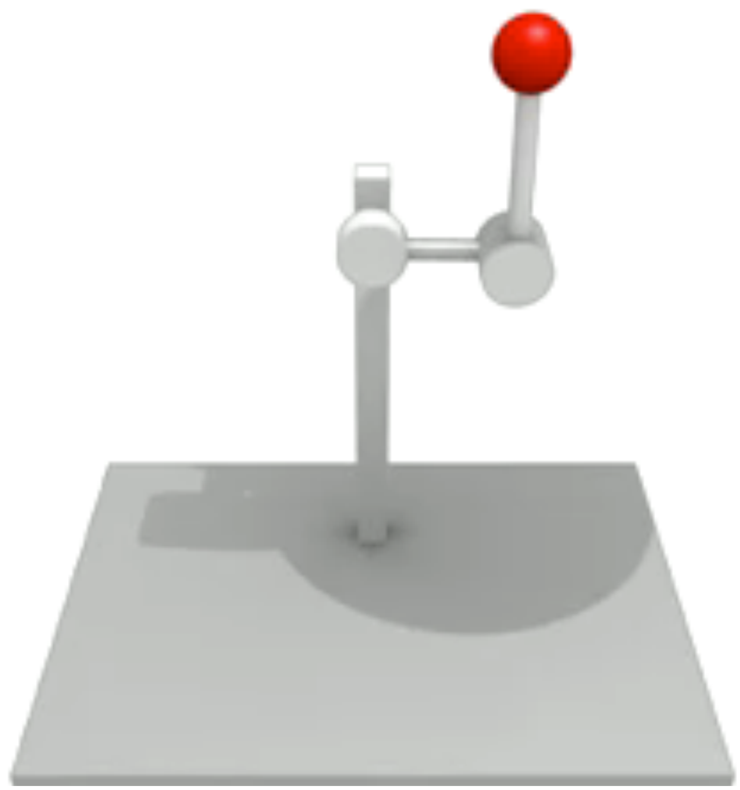
Chaotic attractor

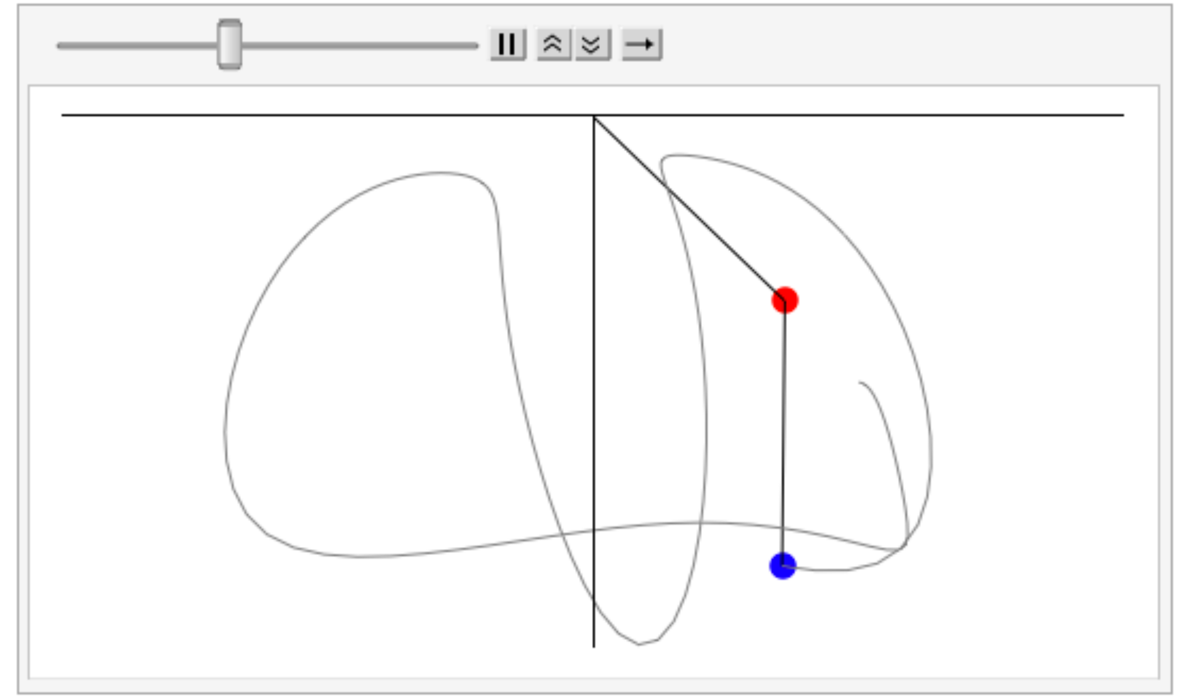
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{bmatrix}$$

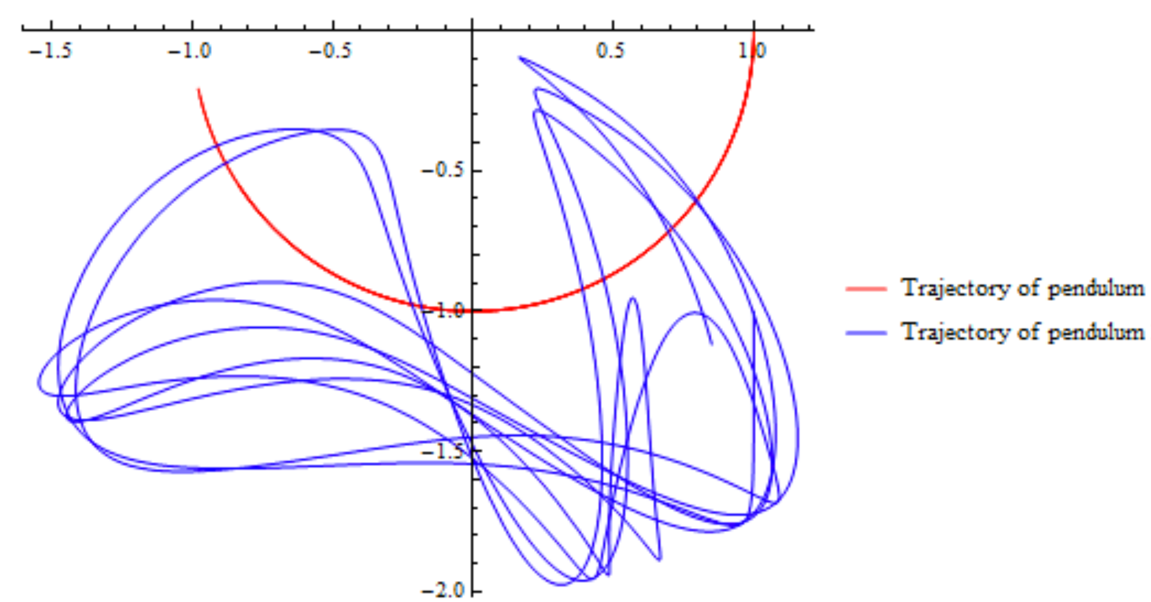


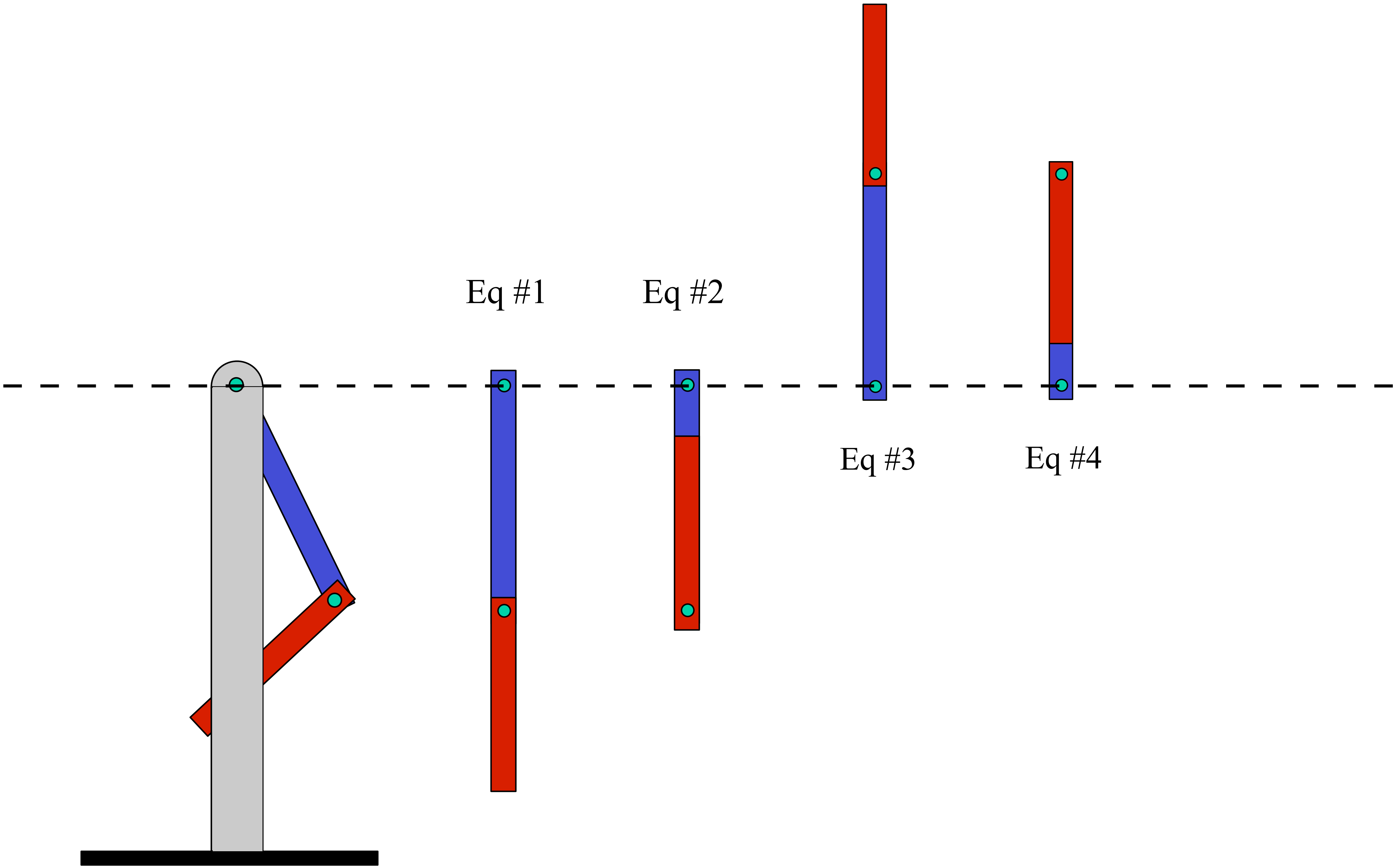
All trajectories converge to $x_e = 0$ but x_e is not stable

Convergence by itself does not imply Stability



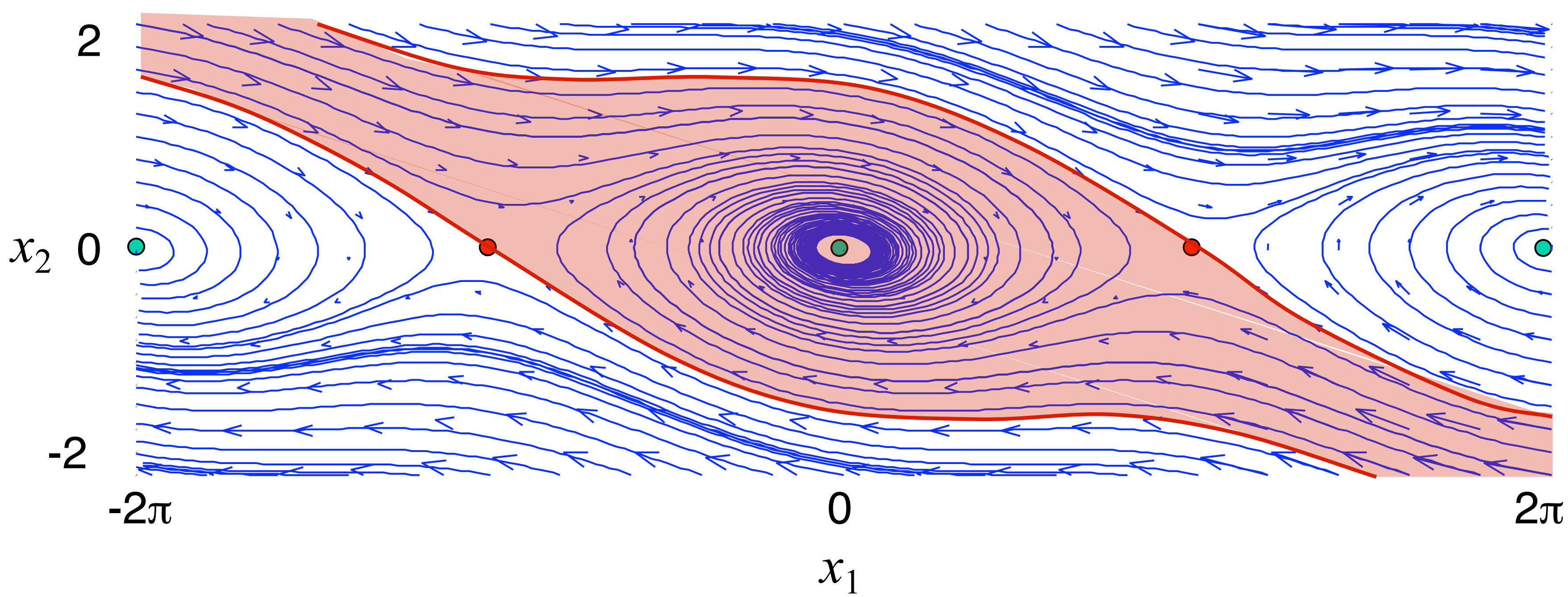



−1.5
−1.0
−0.5
0.5
1.0
−0.5
−1.0
−1.5
−2.0
Trajectory of pendulum 1
Trajectory of pendulum 2


Eq #1
Eq #2
Eq #3
Eq #4

Local v. global behavior

- Stability is a local behavior
  - define local behavior of the system
  - feature of each e.p.
  - system can have stable and unstable e.p.



- Region of attraction
  - initial conditions that converge to e.p.
  - local or global feature

Key questions

- Do equilibrium states exist in the data?
- Are equilibrium states unique?
- Are equilibrium states optimal?
- Are equilibrium states stable?