Lecture 3
- what are equilibria? -

Review

Modeling

- A simplified, quantified representation
- Dynamical system or process (driven by time)
- Answer questions via:
  - formal analysis
  - simulation
- Driven by the rise of big data
- Good model = useful model

Dynamic modeling

deterministic
stochastic

continuous time
(differential equations)

discrete time
(difference equation)

time-driven

discrete event
systems

event-driven

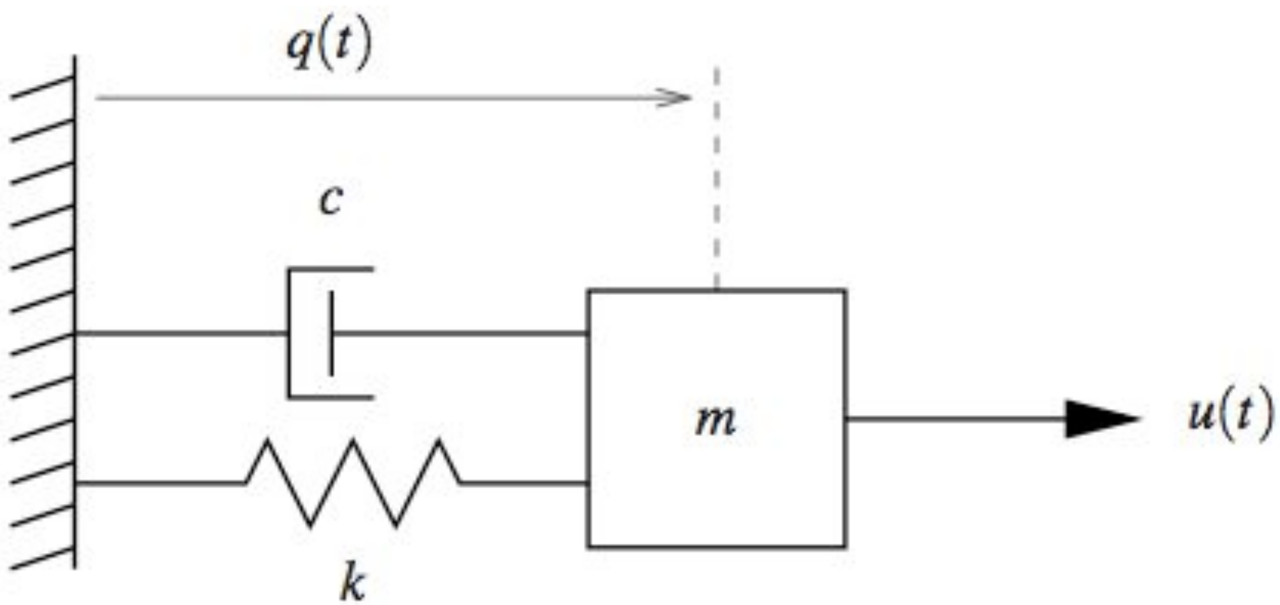
hybrid
systems

Adapted from S. N. Sreenach et al., Modeling the dynamics of signaling pathways, Essays in Biochemestry, vol. 45, 2008.

Central concept in modeling

- The state of the system!
- Dynamic model: state, input, outputs, dynamics
- What it is:
    - Independent quantities that determine future evolution
- What it does:
    - Captures effects of the past

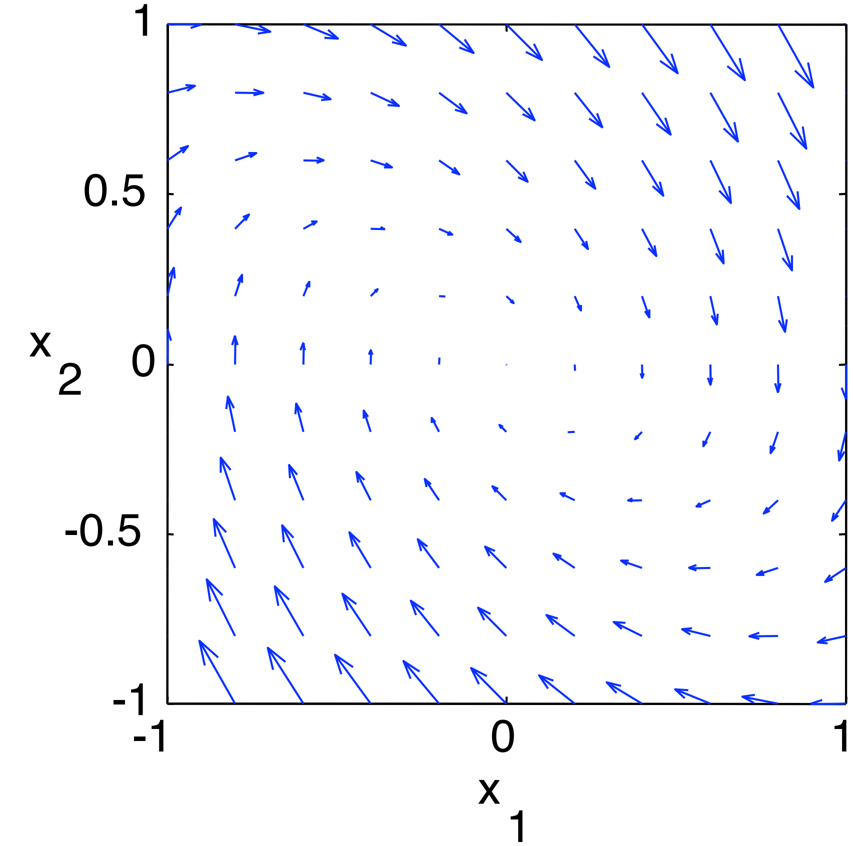
Phase planes

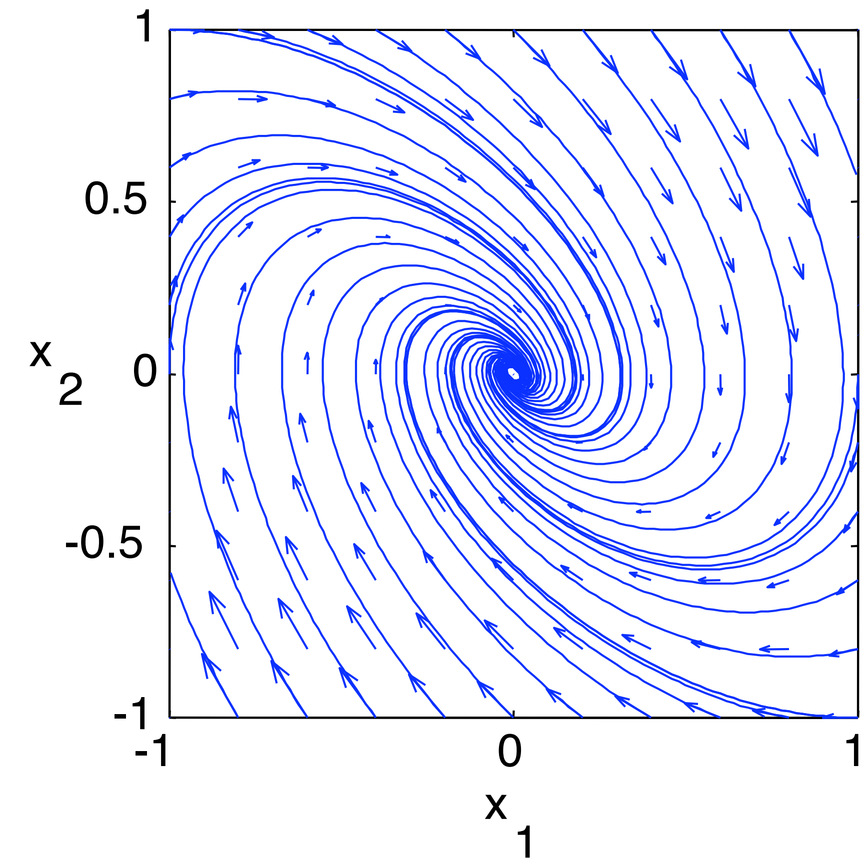


$$m\frac{d^2q}{dt^2} + c\frac{dq}{dt} + kq = 0$$

$$c = 1,\ k = 1,\ m = 1$$

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{bmatrix} \qquad \begin{aligned} x_1 &= q \\ x_2 &= \dot{q} \end{aligned}$$





Today

- ODEs for (state space) modeling
- Linear v. nonlinear systems
- Equilibrium points
- Types of stability
- Local v. global properties
- Why care about stability?

Linear v. nonlinear systems

Differential equations

General form
$$\begin{cases} \dfrac{dx}{dt} &= f(x,u) \\ y &= h(x) \end{cases}$$

First order
Linear systems
$$\begin{cases} \dfrac{dx}{dt} &= ax + bu \\ y &= cx + du \end{cases}$$

What about higher order linear differential equations?

$$\frac{d^n q}{dt^n} + a_1 \frac{d^{n-1} q}{dt^{n-1}} + \ldots + a_n q = u$$
$$y = b_1 \frac{d^{n-1} q}{dt^{n-1}} + \ldots + b_{n-1} \frac{dq}{dt} + b_n q$$

$$\Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d^{n-1}q/dt^{n-1} \\ d^{n-2}q/dt^{n-2} \\ \vdots \\ dq/dt \\ q \end{bmatrix}$$

$$\Downarrow$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \ldots & -a_{n-1} & -a_n \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = [b_1 \quad b_2 \quad \ldots \quad b_n] x$$

Linear systems $\left\{ \begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \end{aligned} \right.$

x = state; n^{th} order, $x \in \mathbb{R}^n$

u = input; $x \in \mathbb{R}^p$

y = output; y∈R^q

Difference equations

General form
$$\begin{cases} x_{k+1} &= f(x_k, u_k) \\ y_k &= h(x_k) \end{cases}$$

Linear systems
$$\begin{cases} x_{k+1} &= A_d x_k + B_d u_k \\ y_k &= C_d x_k \end{cases}$$

$$A_d = e^{AT} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}_{t=T}$$

$$B_d = \left(\int_{\tau=0}^{T} e^{A\tau} d\tau \right) B$$

$$= A^{-1}(A_d - I)B \text{ if } A \text{ is nonsingular}$$

$$C_d = C$$

Continuous differential equations into discrete difference equations!

ODE with derivative input

Consider

$$\ddot{w} + 5\dot{w} + 6w = \ddot{r} + \dot{r} + 2r$$

Let

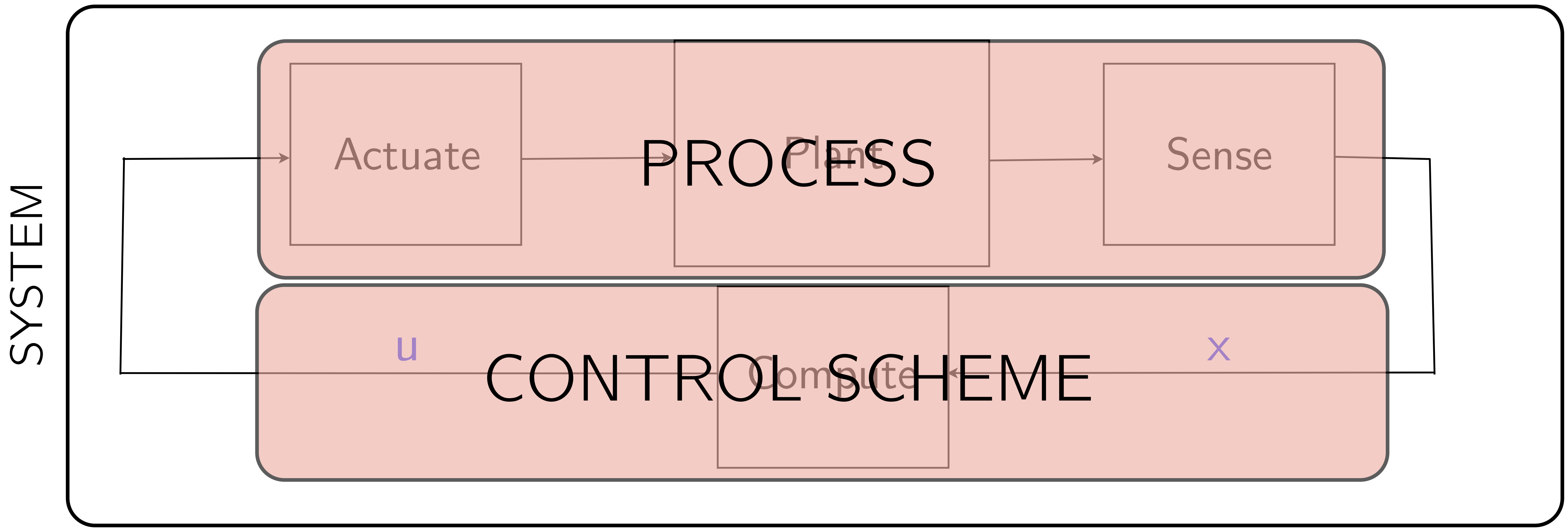
$$\begin{aligned} u &= r \\ x_1 &= w - \beta_0 u \\ x_2 &= \dot{w} - \beta_1 u - \beta_0 \dot{u} \end{aligned}$$

Then

$$\begin{aligned} \dot{x}_1 &= \dot{w} - \beta_0 \dot{u} \\ &= x_2 + \beta_1 u \\ \dot{x}_2 &= \ddot{w} - \beta_1 \dot{u} - \beta_0 \ddot{u} \\ &= -5\dot{w} - 6w + \ddot{u} + \dot{u} + 2u - \beta_1 \dot{u} - \beta_0 \ddot{u} \end{aligned}$$

Letting $\beta_0 = 1,\ \beta_1 = -4$

$$\begin{aligned} \dot{x}_2 &= -5(\dot{w} - \dot{u}) - 6w + 2u \\ &= -5(x_2 - 4u) - 6(x_1 + u) + 2u \\ &= -6x_1 - 5x_2 + 16u \end{aligned}$$

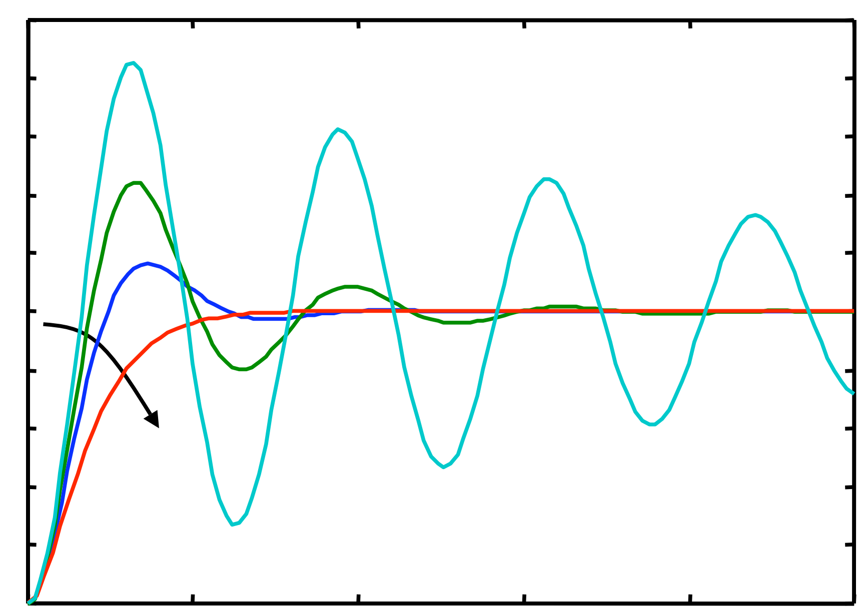


$$\dot{x} = f(x, u) \qquad u = \alpha(x)$$

plant input

control law

- Stability: closed-loop stable?
- Performance: closed-loop behavior?
- Robustness (later)



Calling something a system does not make it stable, controllable, or even analyzable...

Equilibrium points

e•qui•lib•ri•um |ˌēkwəˈlibrēəm; ˌekwə-|

noun (pl. **-lib•ri•a** |-ˈlibrēə|)

a state in which opposing forces or influences are balanced *: the maintenance of social equilibrium.*

• a state of physical balance *: I stumbled over a rock and recovered my equilibrium.*

• a calm state of mind *: his intensity could unsettle his equilibrium.*

• Chemistry a state in which a process and its reverse are occurring at equal rates so that no overall change is taking place *: ice is in **equilibrium with** water.*

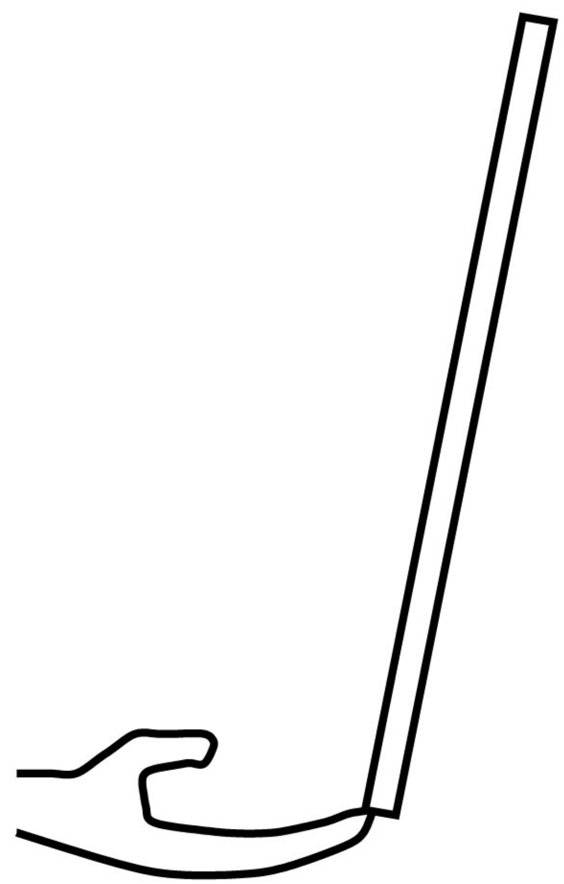
• Economics a situation in which supply and demand are matched and prices stable.

DERIVATIVES

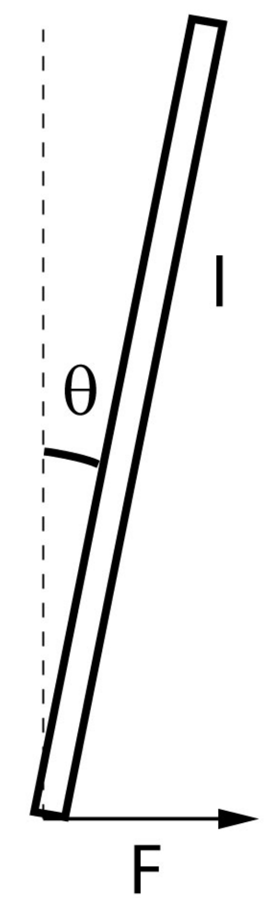
e•qui•lib•ri•al |-ˈlibrēəl| adjective

Example: Inverted pendulum

Segway



Balancing a stick



Inverted pendulum

Stationary conditions for the pendulum

Given

$$\dot{x} = f(x)$$

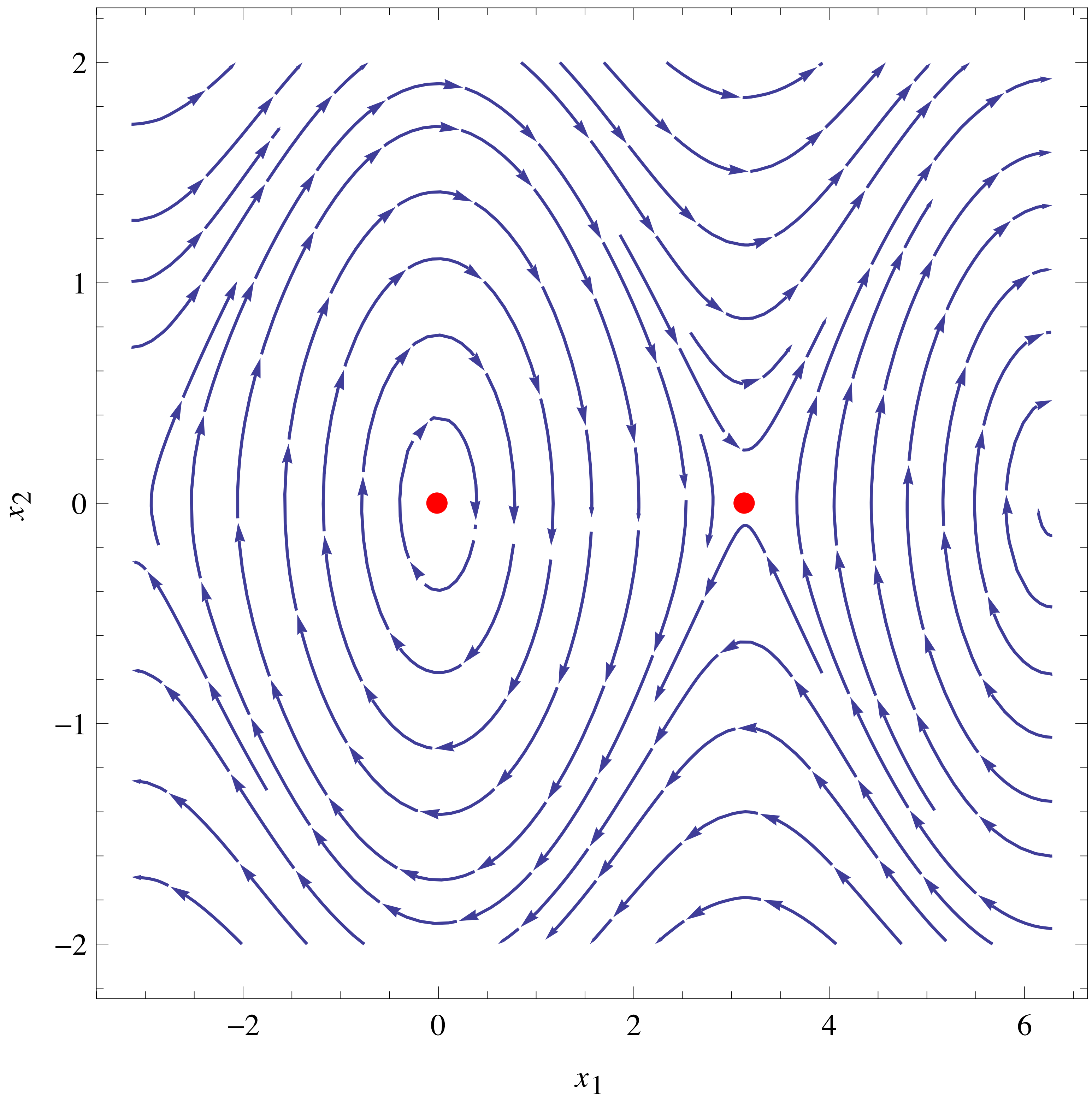
find states such that

$$f(x_e) = 0$$

No-friction case

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 \end{bmatrix}$$

$$x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}$$



We can always assume $x_e = 0$

Consider $\dot{x} = f(x)$, $f(x_e) = 0$, with an equilibrium $x_e \neq 0$

$$z = x - x_e$$

$$\dot{z} = \dot{x} = f(x)$$

$$f(x) = f(z + x_e) := g(z)$$

Note that because

$$g(0) = f(0 + x_e) = f(x_e) = 0$$

The equilibrium z_e of the new system $\dot{z} = g(z)$ is $z_e = 0$.

e.g.