Lecture 18

- how to meet system specifications? -

Review

System specifications

- Percentage of overshoot

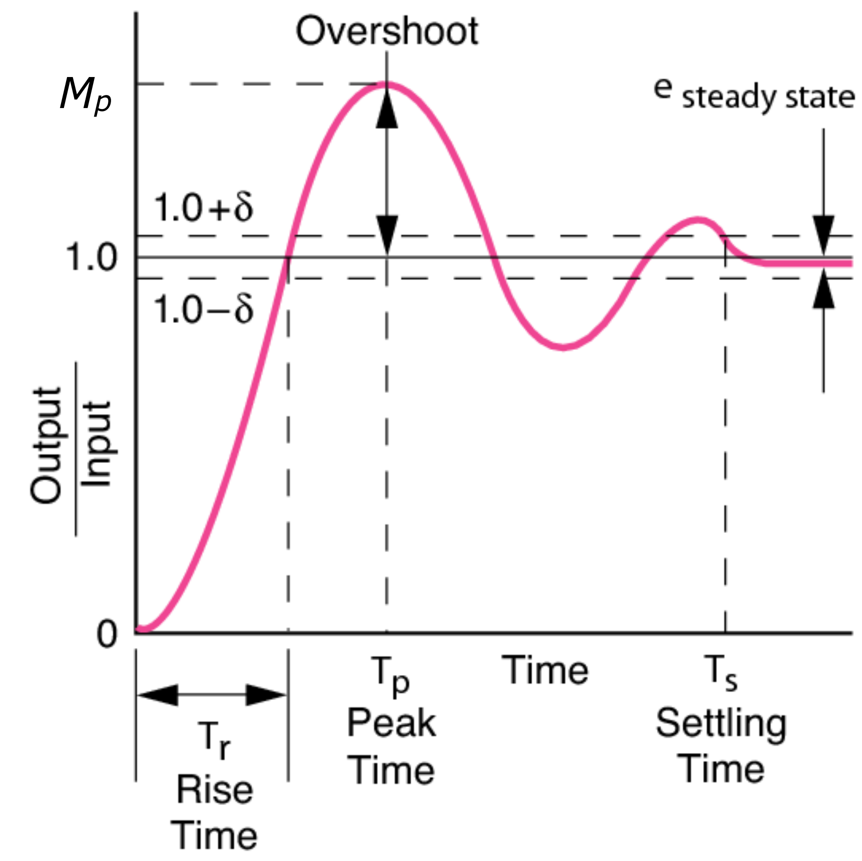
$$P.O. = 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

- Peak time

$$T_p = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}}$$

- Settling time

$$T_s \approx \frac{4}{\zeta\omega_0}$$



- Steady state error

$$e_{ss} = \text{reference value} - \text{final value}$$

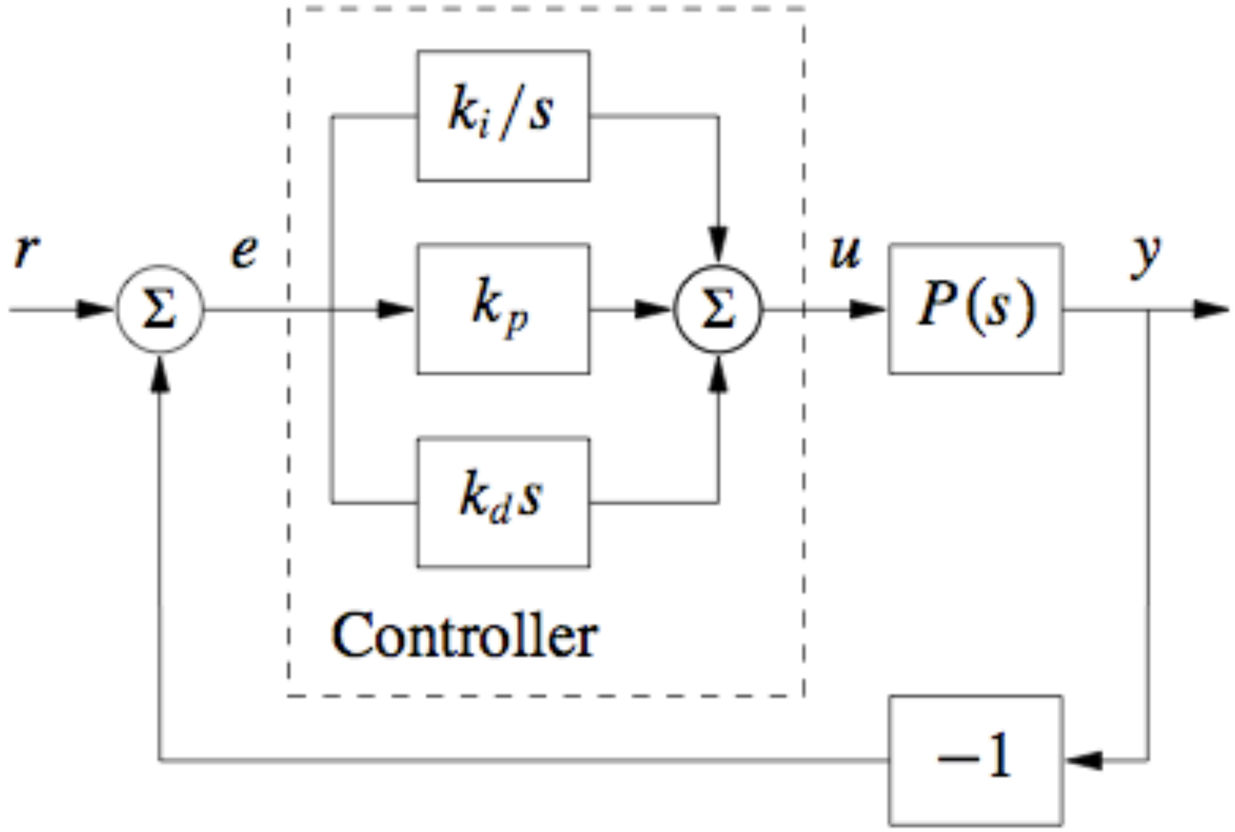
- Stability margins

$$g_m = 2 - 5 \text{ and } \phi_m = 30^\circ - 60^\circ$$

Today

- PID (proportional–integral–derivative controller)
- Basic properties of each component
- PI implementation
- How to meet system specifications? (if possible)

Structure of PID controller



$$u = k_p e + k_i \int_0^t e(\tau) d\tau + k_d \frac{de}{dt} = k_p \left(e + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de}{dt} \right)$$

- Most common way of using feedback in engineering systems
- About 97% of all controllers are of type PID
- Ideal parallel form

Transfer function of PID

- Ideal parallel form

$$u = k_p e + k_i \int_0^t e(\tau) d\tau + k_d \frac{de}{dt} = k_p \left(e + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de}{dt} \right)$$

- Laplace form

$$\begin{aligned} u &= k_p e + \frac{k_p}{T_i} \int_0^t e(\tau) d\tau + k_p T_d \frac{de}{dt} \\ &= k_p e + k_i \int_0^t e(\tau) d\tau + k_d \frac{de}{dt} \end{aligned}$$

- Transfer function

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s}$$

Variations of the PID

- PID controller

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s}$$

- P controller

$$C(s) = k_p$$

: pure proportional controller (simple gain)

- PI controller

$$C(s) = \frac{k_p s + k_i}{s}$$

: adds a pole at the origin

- PD controller

$$\begin{aligned} C(s) &= \frac{k_d s^2 + k_p s}{s} \\ &= k_d s + k_p \\ &= k_d\left(s + \frac{k_p}{k_d}\right) \end{aligned}$$

: adds a zero at $-k_p/k_d$

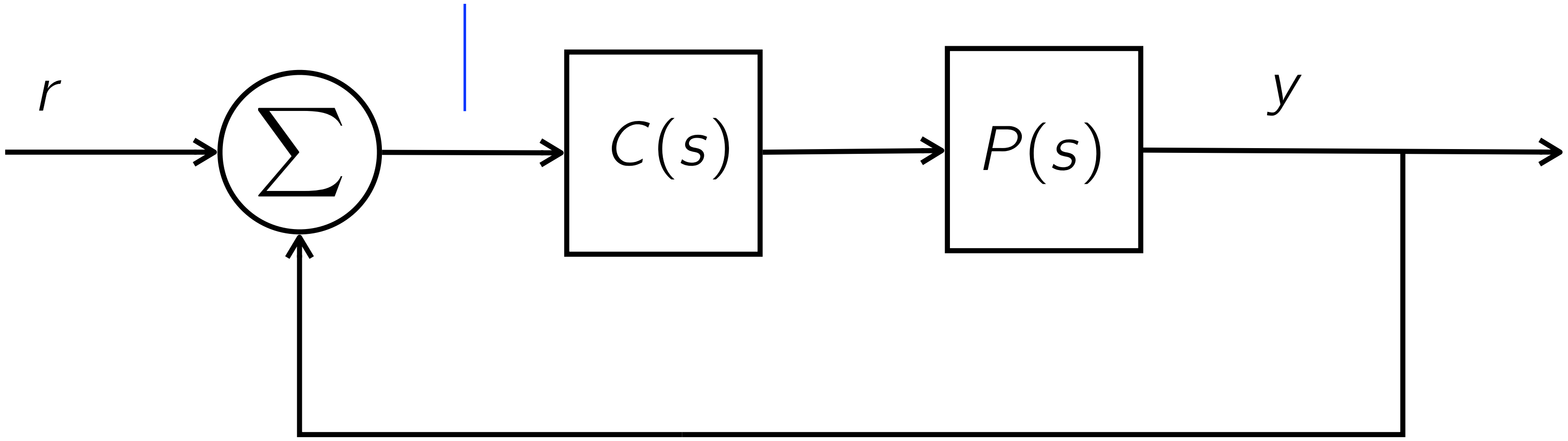
Problem

Consider a system with the following open loop transfer function:

$$P(s) = \frac{0.25}{s + 0.1}$$

$$C(s) = k_p$$

e_{ss} = reference value − final value



Compute the steady-state error (when the reference signal is a unit step)

Problem

- Using properties of Laplace

$$
\begin{aligned}
e_{ss} &= \lim_{t\to\infty} [r(t) - y(t)] \\
&= \lim_{s\to 0} s\,[R(s) - Y(s)]
\end{aligned}
$$

- For a unit step

$$R(s) = \frac{1}{s}$$

In open-loop

$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} s\left[\frac{1}{s} - C(s)P(s)\frac{1}{s}\right] \\
&= \lim_{s\to 0}\left[1 - k_p\frac{0.25}{s+0.1}\right] = 1 - 2.5k_p
\end{aligned}
$$

In closed-loop

$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0}\left[1 - \frac{C(s)P(s)}{1 + C(s)P(s)}\right] \\
&= \lim_{s\to 0}\left[1 - \frac{0.25k_p}{s + 0.1 + 0.25k_p}\right] = 1 - \frac{0.25k_p}{0.1 + 0.25k_p}
\end{aligned}
$$

Remarks

- Zero state state error requires
  - Open-loop: $k_p = 0.4$ (not practical)
  - Closed-loop: k_p tends to infinity (not possible)

- Consider a PI controller

$$C(s) = \frac{k_p s + k_i}{s}$$

- Key questions:
  - What is the steady state error with a PI controller?
  - What value of k_i is required?

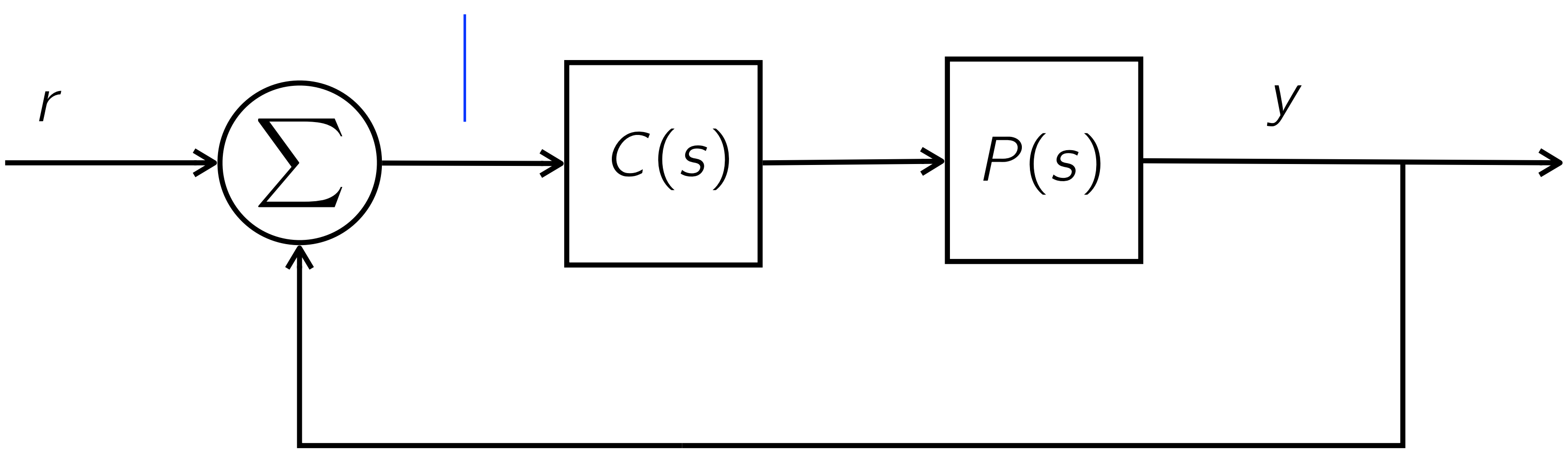
Problem

Consider a system with the following open loop transfer function:

$$P(s) = \frac{0.25}{s + 0.1}$$

$$C(s) = \frac{k_p s + k_i}{s}$$

$e_{ss} = \text{reference value} - \text{final value}$



Compute the steady-state error (when the reference signal is a unit step)

Steady state error for PI

- Compute open-loop transfer function

$$C(s)P(s) = \frac{k_p s + k_i}{s} \frac{0.25}{s + 0.1}$$

- Steady state error (open-loop)

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} [1 - C(s)P(s)] \\ &= \lim_{s \to 0} \left[1 - \frac{k_p s + k_i}{s} \frac{0.25}{s + 0.1} \right] \\ &= -\infty \end{aligned}$$

Steady state error grows to infinity!

Steady state error for PI

- Compute closed-loop transfer function

$$\frac{C(s)P(s)}{1+C(s)P(s)} = \frac{\frac{k_p s+k_i}{s}\frac{0.25}{s+0.1}}{1+\frac{k_p s+k_i}{s}\frac{0.25}{s+0.1}}$$
$$= \frac{0.25(k_p s+k_i)}{s(s+0.1)+0.25(k_p s+k_i)}$$

- Steady state error (closed-loop)

$$e_{ss} = \lim_{s\to 0}\left[1-\frac{C(s)P(s)}{1+C(s)P(s)}\right]$$
$$= \lim_{s\to 0}\left[1-\frac{0.25(k_p s+k_i)}{s(s+0.1)+0.25(k_p s+k_i)}\right]$$
$$= 1-\frac{k_i}{k_i}$$

Zero steady state error for any $k_i \neq 0$

Problem

Consider a system with the following open loop transfer function:

$$P(s) = \frac{0.25}{s + 0.1}$$

$$C(s) = \frac{k_p s + k_i}{s}$$

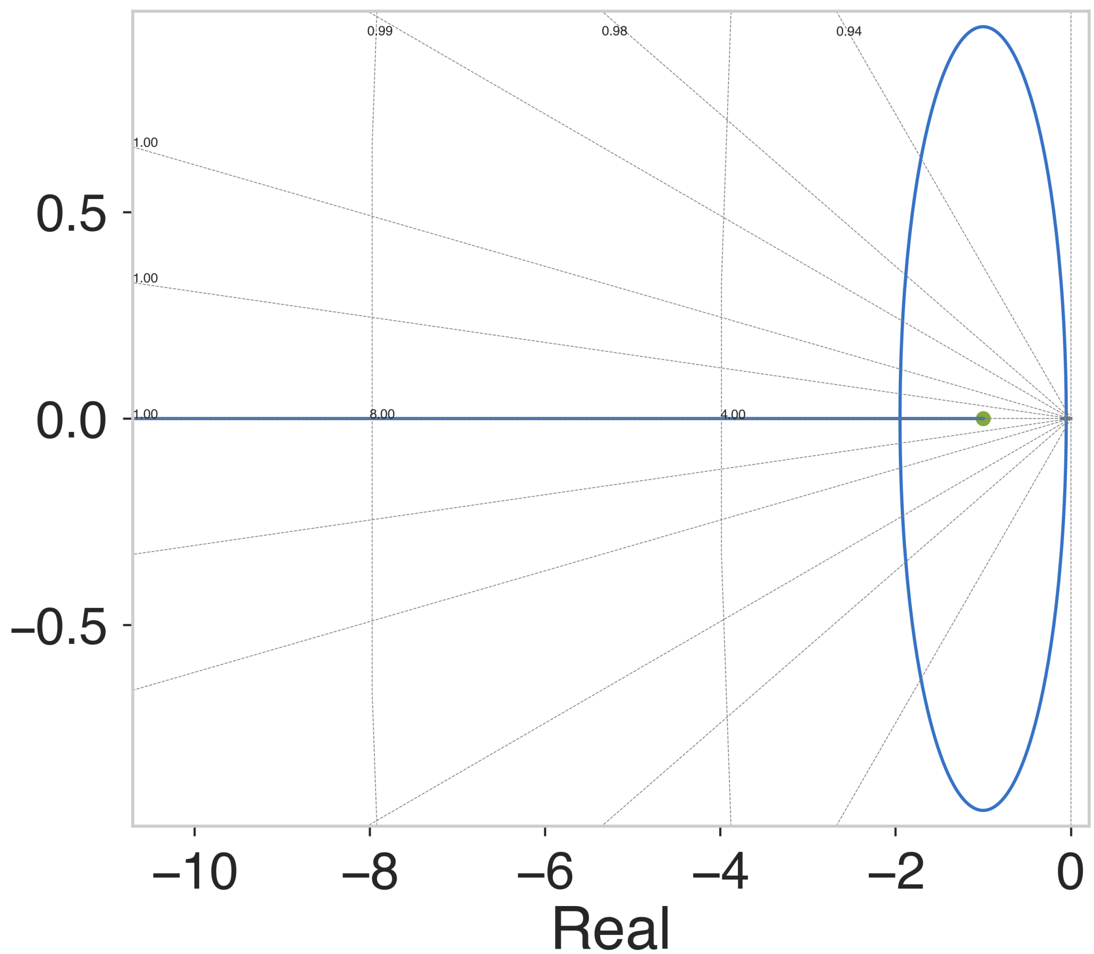
Through simulations:

- Validate the steady-state error
- Compute the gain and phase margin
- Compute the settling time T_s

How can you make the system reach its steady state faster?

Problem

- Look at root locus
- For higher gains, the poles move away from the imaginary axis



Problem

- Increasing the value of k_p does not always work
- Consider the system

$$P(s) = \frac{1}{s^2 + 2s}$$

$$C(s) = \frac{k_p s + k_i}{s}$$

Need to implement a PD controller!